

# UNIQUE CONTINUATION FROM INFINITY IN ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES II: NON-STATIC BOUNDARIES

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**ABSTRACT.** We generalize our unique continuation results recently established for a class of linear and nonlinear wave equations  $\square_g \phi + \sigma \phi = \mathcal{G}(\phi, \partial \phi)$  on asymptotically anti-de Sitter (aAdS) spacetimes to aAdS spacetimes admitting non-static boundary metrics. The new Carleman estimates established in this setting constitute an essential ingredient in proving unique continuation results for the full nonlinear Einstein equations, which will be addressed in forthcoming papers. Key to the proof is a new geometrically adapted construction of foliations of pseudoconvex hypersurfaces near the conformal boundary.

## 1. INTRODUCTION

In [7], we initiated the study of unique continuation properties of  $(n+1)$ -dimensional asymptotically anti-de Sitter (aAdS) spacetimes  $(\mathcal{M}, g)$  by studying a class of tensorial linear and non-linear Klein-Gordon equations,

$$(1.1) \quad \square_g \phi + \sigma \phi = \mathcal{G}(\phi, \partial \phi),$$

in a portion of spacetime near the conformal boundary  $\mathcal{I}$ , with  $\sigma \in \mathbb{R}$  and suitable assumptions on  $\mathcal{G}(\phi, \partial \phi)$ .

The spacetimes  $(\mathcal{M}, g)$  considered in equation (1.1) encompassed a large class of Lorentzian metrics, not necessarily Einstein-vacuum, including in particular non-stationary spacetimes. The main restriction in [7] was the assumption that the  $n$ -dimensional Lorentzian metric  $\mathring{g}$  induced by  $g$  on the boundary  $\mathcal{I}$  (after conformal transformation) was *static*. In this paper, we will remove this assumption and extend the unique continuation results of [7] to a class of metrics which are *not* required to be static on the boundary.

Our main motivation originates from general relativity, where spacetimes with non-static boundary metrics appear naturally by solving an initial boundary value problem for the vacuum Einstein equations<sup>1</sup>

$$(1.2) \quad Ric[g] = \Lambda g = -\frac{n(n-1)}{2\ell^2}g.$$

Indeed, in dimension 3+1, Friedrich [3] constructed a large class of aAdS spacetimes satisfying (1.2) for which the conformal class of the  $n$ -dimensional metric on the boundary can be freely prescribed a priori. A particularly interesting case arises from so-called dissipative boundary conditions. Here the resulting spacetime will generally not only possess a non-static boundary metric but also exhibit a non-trivial flux of gravitational radiation through its boundary. See also [6].

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<sup>1</sup>Here and in the rest of the paper, we choose the normalization  $\ell^2 = 1$  for convenience.

In view of the above, extending the Carleman estimates of [7] to spacetimes with general (dynamical) boundary metrics ensures that the class of metrics satisfying (1.2) for which a unique continuation property holds agrees with the class of metrics that arises naturally from the forward initial boundary value problem for (1.2). This is a prerequisite for proving unique continuation results for the non-linear Einstein equations in a sensible class. We finally remark that the class of spacetimes considered here is also natural in the context of the AdS/CFT correspondence [8].

**1.1. The Class of aAdS Spacetimes.** We first give an informal definition of the class of aAdS spacetimes to be considered. Unlike in [7], we will in this paper exhibit these spacetimes in Fefferman–Graham (FG) coordinate systems near the boundary. Such coordinates are well-adapted to the geometric problem at hand and simplify many of the computations. We note that this does not constitute any loss of generality; in Appendix A, we demonstrate how to transform a metric in the coordinates used in [7] to a metric in FG form.

Specifically, we consider manifolds  $\mathcal{M} = (0, \rho_0) \times (T_-, T_+) \times \mathcal{S}$ , with  $\mathcal{S}$  an  $(n-1)$ -dimensional Riemannian manifold, and with the local coordinates of  $\mathcal{S}$  denoted collectively by  $x$ . We will equip  $\mathcal{M}$  with metrics of the form

$$(1.3) \quad g = \frac{d\rho^2 + \mathfrak{g}(\rho)}{\rho^2},$$

where  $\mathfrak{g}$  is a family of Lorentzian metrics on the level sets of  $\rho$  with the expansion

$$(1.4) \quad \mathfrak{g}(\rho, t, x) = \mathring{\mathfrak{g}}(t, x) + \rho^2 \bar{\mathfrak{g}}(t, x) + \rho^3 \hat{\mathfrak{g}}(t, x) + \mathcal{O}(\rho^3).$$

Here  $\mathring{\mathfrak{g}}$ ,  $\bar{\mathfrak{g}}$ , and  $\hat{\mathfrak{g}}$  are tensors defined on the level sets of  $\rho$  whose components are independent of the particular level set chosen.<sup>2</sup>

As is well-known [2, 4], if  $g$  satisfies (1.2) and  $n \geq 3$ , then  $-\bar{\mathfrak{g}}$  coincides with the Schouten tensor  $\hat{P}$  of  $\mathring{\mathfrak{g}}$ , cf. also Appendix B. Furthermore, in dimension  $n > 3$ , the tensor  $\hat{\mathfrak{g}}$  is also determined by  $\mathring{\mathfrak{g}}$ , while for  $n = 3$ ,  $\hat{\mathfrak{g}}$  is the “stress-energy tensor” on the boundary, which is not determined by  $\mathring{\mathfrak{g}}$  but by the full spacetime Weyl tensor. For example, for Schwarzschild-AdS spacetimes in  $n = 3$ , one computes

$$\mathring{\mathfrak{g}} = -dt^2 + \tilde{\gamma}, \quad \bar{\mathfrak{g}} = -\frac{1}{2}(dt^2 + \tilde{\gamma}), \quad \hat{\mathfrak{g}} = \frac{2}{3}M(2dt^2 + \tilde{\gamma}),$$

where  $\tilde{\gamma}$  is the round metric on the unit sphere. For pure AdS spacetime, the above identities hold, but with  $\hat{\mathfrak{g}} \equiv 0$ .

The metrics of interest in this paper are precisely those satisfying  $\mathcal{L}_{\partial_t} \mathring{\mathfrak{g}} \neq 0$ , while in [7], we assumed  $\mathcal{L}_{\partial_t} \mathring{\mathfrak{g}} = 0$ . Note also that in the important special case of Einstein-vacuum metrics arising from small perturbations of stationary aAdS spacetimes,  $\mathcal{L}_{\partial_t} \mathring{\mathfrak{g}}$  is expected to be small in a suitable norm.

**1.2. Previous results.** We turn to the unique continuation results for (1.1) on segments  $(\mathcal{M}, g)$  defined above. We first recall from [7] the quantities

$$(1.5) \quad \beta_{\pm} = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - \sigma}$$

associated with the mass  $\sigma$  in (1.1). Precise assumptions on the right hand side  $\mathcal{G}$  in (1.1) will be made below, cf. (1.11) and (1.13).<sup>3</sup>

<sup>2</sup>Theorems 1.1 and 1.2 below only use (1.4) with the last two terms replaced by  $\mathcal{O}(\rho^3)$ .

<sup>3</sup>From the point of view of decay near the boundary, these assumptions will essentially allow us to treat  $\mathcal{G}$  as a perturbation of the linear Klein-Gordon operator on the left hand side.

We next define the *local unique continuation property* that can be established for solutions to (1.1) defined on the segment  $(\mathcal{M}, g)$ . While the full definition is slightly technical, see Definition 5.1, it essentially states that the *local unique continuation property* holds if any classical solution  $\phi$  of (1.1) which satisfies

$$(1.6) \quad \begin{cases} |\rho^{-\beta_+} \phi| + |\nabla_{t,\rho,x}(\rho^{-\beta_++1} \phi)| \rightarrow 0 & \text{if } \sigma \leq \frac{n^2-1}{4}, \\ |\rho^{-(n+1)/2} \phi| + |\nabla_{t,\rho,x}(\rho^{-(n-1)/2} \phi)| \rightarrow 0 & \text{if } \sigma > \frac{n^2-1}{4} \end{cases}$$

on the conformal boundary  $\mathcal{I} = \{\rho = 0\} \times (T_-, T_+) \times \mathcal{S}$  of  $(\mathcal{M}, g)$ , vanishes in an open neighborhood of  $\mathcal{I}$ .<sup>4</sup>

An important observation from [7], which demonstrates that the vanishing conditions (1.6) are somewhat natural, is that if  $\frac{n^2}{4} - 1 < \sigma < \frac{n^2}{4}$  in (1.1), then any classical solution  $\phi$  in  $(\mathcal{M}, g)$  which satisfies *both* Dirichlet *and* Neumann conditions at  $\mathcal{I}$  also satisfies (1.6).<sup>5</sup>

The key uniqueness theorem of [7] can now be rephrased in the Fefferman–Graham coordinates introduced above as:

**Theorem 1.1** (cf. Theorem 1.3 and Theorem 4.2 of [7]). *Let*

$$(\mathcal{M} := (0, \rho_0) \times (0, T\pi) \times \mathcal{S}, g)$$

*be an aAdS spacetime segment whose boundary data  $\mathring{\mathbf{g}}$  and  $\bar{\mathbf{g}}$  satisfy that  $\mathring{\mathbf{g}}$  is static on  $\mathcal{I}$  (i.e.,  $\mathcal{L}_{\partial_t} \mathring{\mathbf{g}} = 0$ ), as well as the following pseudoconvexity condition:*

$$(1.7) \quad -\bar{\mathbf{g}} - \frac{1}{T^2} dt^2 - \zeta \mathring{\mathbf{g}} \text{ is positive-definite for some bounded } \zeta \in C^\infty(\mathcal{I}).$$

*Then, the local unique continuation property holds on  $(\mathcal{M}, g)$  for (1.1), provided  $\mathcal{G}(\phi, \partial\phi)$  satisfies the estimate*

$$(1.8) \quad |\mathcal{G}(\phi, \partial\phi)|^2 \leq C \rho^p (\rho^4 |\nabla_{t,\rho,x} \phi|^2 + \rho^{2p} |\phi|^2)$$

*in  $(\mathcal{M}, g)$  for some  $p > 0$  and some constant  $C > 0$ .*

We remark that for  $(\mathcal{M}, g)$  being a segment of the exact AdS spacetime with cosmological constant  $\Lambda = -\frac{n(n-1)}{2}$ , any  $T > 1$  will guarantee that condition (1.7) holds, while the condition cannot be satisfied if  $T \leq 1$ . The borderline case  $T = 1$  (i.e., the segment having time length  $1 \cdot \pi$ ) corresponds precisely to the (re)focusing time of null geodesics emanating from the boundary, cf. Figure 1. As explained in [7], in view of the counterexamples of [1], this restriction on the time length of the segment seems necessary. Cf. also Theorem 1.3 below.

More generally, one sees that if  $g$  in Theorem 1.1 is Einstein-vacuum, then a boundary metric  $\mathring{\mathbf{g}}$  with positive Schouten tensor  $\mathring{P} = -\bar{\mathbf{g}}$  will ensure condition (1.7) holds for large enough  $T$ , the optimal  $T$  being closely related to the refocusing time of null geodesics near the boundary. See Appendix B for quantitative statements.

One of the key difficulties in proving Theorem 1.1 derives from the fact that the conformal boundary is only zero-pseudoconvex, and hence standard unique continuation results fail. The proof in [7] constructed a foliation of the spacetime segment by pseudoconvex hypersurfaces, whose existence in turn depended crucially on the

<sup>4</sup>The proper Definition 5.1 stipulates slightly weaker vanishing conditions in  $L^2$ . Furthermore, the vanishing condition for  $\nabla_x \phi$  in (1.6) is in fact not necessary and is replaced in Definition 5.1 by a weaker finite integral condition (which is a consequence of finite energy).

<sup>5</sup>Recall that only for the aforementioned  $\sigma$ -range does one have the freedom of specifying boundary conditions for the forward boundary initial value problem. See the discussion in [7].

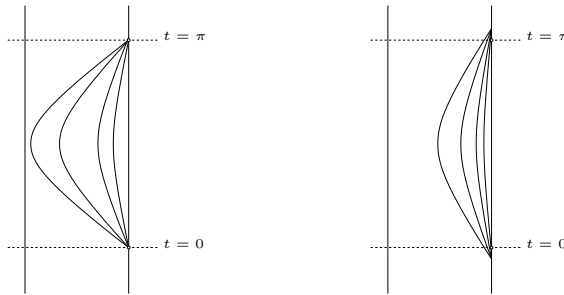


FIGURE 1. Illustration of the refocusing of null geodesics (left) and the pseudoconvex foliation (right) in pure AdS.

time-length of the segment and eventually led to condition (1.7). From the foliation (depicted schematically for pure AdS in Figure 1), we deduced after suitable renormalization a Carleman estimate which implied the unique continuation property stated in Theorem 1.1. We also emphasize that, with applications to general relativity in mind, we actually proved both the Carleman estimates and the uniqueness theorems for a class of *tensorial* wave equations in [7].

**1.3. The main result.** We turn to the main result of this paper, which generalizes Theorem 1.1 by removing the staticity assumption  $\mathcal{L}_{\partial_t} \mathring{\mathbf{g}} = 0$  and replacing condition (1.7) appropriately. As in [7], the main technical difficulty is to define a foliation of pseudoconvex timelike hypersurfaces near the boundary for this class of spacetimes. This requires a new idea, because, as it turns out, even if the  $\mathcal{L}_{\partial_t} \mathring{\mathbf{g}} \neq 0$  is small, the extra terms arising from this cannot be treated perturbatively: pseudoconvex hypersurfaces defined for a spacetime satisfying  $\mathcal{L}_{\partial_t} \mathring{\mathbf{g}} = 0$  will in general cease to be pseudoconvex if the metric is perturbed such that  $\mathcal{L}_{\partial_t} \mathring{\mathbf{g}} \neq 0$ .

We resolve this problem by starting from a general ansatz for the level sets of the foliation, which eventually connects pseudoconvexity of the level sets to the existence of particular solutions to an ordinary differential inequality (ODI), whose coefficients depend on  $\mathcal{L}_{\partial_t} \mathring{\mathbf{g}}$ . Very schematically, in the boundary-static case, this ODI is simply a harmonic oscillator-type ODE. In the dynamic case, one is instead led to a damped harmonic oscillator, with the damping term determined by  $\mathcal{L}_{\partial_t} \mathring{\mathbf{g}}$ .<sup>6</sup>

Once the foliation has been defined, the proof of the Carleman estimate and the uniqueness statements proceed as in [7] apart from minor technical difficulties which will be discussed in the bulk of the paper. This leads to the following rough version of our main result; see Section 5 for the precise statement.

**Theorem 1.2.** *Let  $(\mathcal{M} := (0, \rho_0) \times (0, T\pi) \times \mathcal{S}, g)$  be an aAdS spacetime segment whose boundary data<sup>7</sup>  $\mathring{\mathbf{g}} = -dt^2 + \gamma_{AB}(t, x) dx^A dx^B$  and  $\mathring{\mathbf{g}}$  satisfy the following pseudoconvexity conditions:*

- *There exists  $\xi > 0$  such that for any vector fields  $X, Y$  tangent to  $\mathcal{S}$ ,*

$$(1.9) \quad |\mathcal{L}_{\partial_t} \gamma(X, Y)| \leq \xi \cdot \gamma(X, Y).$$

<sup>6</sup>As in the boundary-static case, one should eventually be able to relate the resulting ODI to (approximate) null geodesics on these spacetimes. However, we will not pursue this here.

<sup>7</sup>For the assumptions implicit in the form of the metric  $\mathring{\mathbf{g}}$  see the remark below Definition 2.5.

- There is some bounded  $\zeta \in C^\infty(\mathcal{I})$  such that

$$(1.10) \quad -\bar{\mathfrak{g}} - \frac{1}{\tau^2} dt^2 - \zeta \mathring{\mathfrak{g}} \text{ is positive-definite,}$$

where  $\tau > T$  is a constant determined uniquely in terms of  $T$  and  $\xi$ .

Then the local unique continuation property holds on  $(\mathcal{M}, g)$  for (1.1), provided  $\mathcal{G}$  satisfies for some  $p > 0$  and  $C > 0$  the estimate

$$(1.11) \quad |\mathcal{G}(\phi, \partial\phi)|^2 \leq C\rho^p(\rho^4|\nabla_{t,\rho,x}\phi|^2 + \rho^{2p}|\phi|^2).$$

We remark that there is a simple explicit formula for  $\tau$ ; see (3.4) and (3.25). One can also check that for small perturbations of the pure AdS boundary, i.e. for  $\mathcal{L}_{\partial_t}\mathring{\mathfrak{g}}$  small and  $\bar{\mathfrak{g}}, \mathring{\mathfrak{g}}$  close to their pure AdS values, there is a  $T > 1$  close to 1 (the closeness depending on the size of the perturbation) which ensures that conditions (1.9) and (1.10) are indeed satisfied, cf. Section 3.2.3.

**1.4. The borderline case.** The slightly more geometric approach taken in this paper also reveals an interesting new result for the special case of *static* boundaries addressing the “borderline case”  $T = 1$  discussed below Theorem 1.1:

**Theorem 1.3.** *Let  $(\mathcal{M} = (0, \rho_0) \times (0, \pi) \times \mathcal{S}, g)$  be an aAdS spacetime segment whose boundary data satisfy  $\mathring{\mathfrak{g}} = -dt^2 + \tilde{\gamma}$  and  $\bar{\mathfrak{g}} = -\frac{1}{2}(dt^2 + \tilde{\gamma})$ , as well as the following pseudoconvexity condition:*

$$(1.12) \quad -\hat{\mathfrak{g}} - \zeta \mathring{\mathfrak{g}} \text{ is positive-definite for some bounded } \zeta \in C^\infty(\mathcal{I}).$$

Then the unique continuation property holds on  $(\mathcal{M}, g)$  for (1.1), provided  $\mathcal{G}$  satisfies for some  $p > 0$  and  $C > 0$  the estimate

$$(1.13) \quad |\mathcal{G}(\phi, \partial\phi)|^2 \leq C\rho^p(\rho^5|\nabla_{t,\rho,x}\phi|^2 + \rho^{2p}|\phi|^2).$$

Observe that the condition (1.13) on  $\mathcal{G}$  is more stringent than (1.11). This is because the pseudoconvexity of the foliation degenerates faster toward the conformal boundary in Theorem 1.3 than in Theorem 1.2. The proof of Theorem 1.3 logically proceeds along the same lines as the one of Theorem 1.2 but requires a significant refinement of the foliation used in [7] to exploit the higher-order pseudoconvexity.

Note that if  $\hat{\mathfrak{g}}$  is negative definite, then the pseudoconvexity condition holds with  $\zeta = 0$ . In particular, a patch of the  $(3+1)$ -dimensional AdS-Schwarzschild spacetime with negative mass provides a simple example of a spacetime satisfying the assumptions of Theorem 1.3.

**1.5. Overview.** We have written the paper to be essentially self-contained, although [7] contains more extensive explanations of the basic concepts and computations. In Section 2, we define the manifolds and their differentiable structures, as well as the class of aAdS metrics considered on these manifolds. Appendix A relates this class to the class considered in [7]. In Section 3, we construct the level sets and prove the key pseudoconvexity results relevant for Theorem 1.2. In Section 4, we carry out the analogous computations in the static borderline case relevant for Theorem 1.3 only. Section 5 contains the precise statements and proofs of our main uniqueness results, as well as the main Carleman estimates, which exploit the pseudoconvexity properties of the previous sections. Finally, in Appendix B, we collect a geometric interpretation of the result and some applications in the case of Einstein-vacuum metrics with static boundary metrics.

**1.6. Acknowledgement.** The authors thank Claude Warnick for helpful discussions and sharing his notes on the Fefferman-Graham expansions. Both authors acknowledge support through a grant from the European Research Council.

## 2. ASYMPTOTICALLY ADS SPACETIMES

In this section, we construct the class of asymptotically Anti-de Sitter spacetimes that we will treat in our main results. Informally, we will consider spacetimes whose metrics near infinity are of the form

$$(2.1) \quad g = [r^{-2} + r^{-4}\bar{g}_{\rho\rho} + \mathcal{O}(r^{-5})]dr^2 + \mathcal{O}(r^{-3}) \cdot drdt + \mathcal{O}(r^{-3}) \cdot drdx^A \\ + [-r^2 + \bar{g}_{tt} + \mathcal{O}(r^{-1})]dt^2 + [\bar{g}_{tA} + \mathcal{O}(r^{-1})]dtdx^A \\ + [r^2\bar{g}_{AB} + \bar{g}_{AB} + \mathcal{O}(r^{-1})]dx^A dx^B.$$

These include AdS spacetime and, when  $n \geq 3$ , Schwarzschild-AdS and Kerr-AdS spacetimes [5]. A precise description of these spacetimes, in particular the specific coordinates and nature of the “ $\mathcal{O}(r^k)$ ”-error terms, will be given below.

In contrast to [7], here we opt to express (2.1) in a Fefferman-Graham gauge. Roughly, these are asymptotic expansions of the form

$$(2.2) \quad g = \rho^{-2}d\rho^2 + [-\rho^{-2} + \bar{\mathfrak{g}}_{tt} + \mathcal{O}(\rho)]dt^2 + [\bar{\mathfrak{g}}_{AB} + \mathcal{O}(\rho)]dtdx^A \\ + [\rho^{-2}\bar{\mathfrak{g}}_{AB} + \bar{\mathfrak{g}}_{AB} + \mathcal{O}(\rho)]dx^A dx^B.$$

We stress that any expansion of the form (2.1) can be reduced to one of the form (2.2) via a gauge transformation; this will be demonstrated in Appendix A.

Furthermore, we review the notions of horizontal and mixed tensor fields, introduced in [7], that we will use for our general main results. This generalization will be useful in future works, when we apply these results to tensorial quantities present within the Einstein equation. Finally, we conclude the section by computing asymptotic expansions for various geometric quantities.

**2.1. Construction of the Spacetimes.** In this subsection, we define precisely the class of aAdS spacetimes we will consider throughout this paper. The process is a bit more elaborate than in [7], due to the need to include non-static boundaries.

**2.1.1. Preliminaries.** The first step is to prescribe the spacetime topology. This is done by specifying the topology of its AdS-type boundary at infinity.

**Definition 2.1.** *We define the following manifolds:*

- Boundary cross-section: Let  $\mathcal{S}$  be an  $(n-1)$ -dimensional manifold.
- AdS-type boundary: Fix  $T_- < T_+$ , and let  $\mathcal{I} := (T_-, T_+) \times \mathcal{S}$ .
- aAdS spacetime: Let  $\rho_0 > 0$ , and let  $\mathcal{M} := (0, \rho_0) \times \mathcal{I}$ .

*In addition, we let  $p$  denote the projection on  $\mathcal{M}$  to the first component  $(0, \rho_0)$ , and we let  $t$  denote the projection on both  $\mathcal{M}$  and  $\mathcal{I}$  to the  $(T_-, T_+)$ -component.*

Like in [7], we avoid employing a fully geometrically invariant approach, in that we specify most of our asymptotic assumptions in terms of coordinates.

**Definition 2.2.** *Let  $\varphi = (x^1, \dots, x^{n-1})$  denote a coordinate system on  $\mathcal{S}$ .*

- Let  $\varphi_t := (t, x^1, \dots, x^{n-1})$  denote the coordinates on  $\mathcal{I}$  obtained by transporting  $\varphi$ -coordinates along the  $t$ -component and appending  $t$ .
- Let  $\varphi_{\rho,t} := (\rho, t, x^1, \dots, x^{n-1})$  be the coordinates on  $\mathcal{M}$  obtained by transporting the  $\varphi$ -coordinates along the  $(\rho, t)$ -components and appending  $(\rho, t)$ .

Note that the coordinate vector fields  $\partial_t$  arising from the above transported coordinate systems define a *global* vector field on both  $\mathcal{I}$  and  $\mathcal{M}$ . Similarly, the coordinate vector fields  $\partial_\rho$  define a global vector field on  $\mathcal{M}$ .

**Definition 2.3.** *Let  $\varphi$  be a coordinate system on  $\mathcal{S}$ . From now on, we adopt the following coordinate and indexing conventions:*

- *We use upper-case Latin indices  $A, B, \dots$  to denote  $\varphi$ -coordinate components. Similarly, we use  $x^A, x^B, \dots$  to refer to  $\varphi$ -coordinate functions.*
- *We use lower-case Latin indices  $a, b, \dots$  to denote  $\varphi_t$ -coordinate components. Similarly, we use  $x^a, x^b, \dots$  to refer to  $\varphi_t$ -coordinate functions.*
- *We use lower-case Greek indices  $\alpha, \beta, \dots$  to denote  $\varphi_{\rho,t}$ -coordinate components. Similarly, we use  $x^\alpha, x^\beta, \dots$  to refer to  $\varphi_{\rho,t}$ -coordinate functions.*

Next, we define the asymptotic properties of the error terms we will encounter. As mentioned before, the condition here is stronger than that found in that of [7].

**Definition 2.4.** *Consider a spacetime  $(\mathcal{M}, g)$ , where  $\mathcal{M}$  is as in Definition 2.1. Let  $\zeta \in C(\mathcal{M})$ , and let  $\varphi$  denote a coordinate system on  $\mathcal{S}$ . We use the symbol  $\mathcal{O}_\varphi(\zeta)$  to denote a smooth function  $u$  on an appropriate open subset of  $\mathcal{M}$  (depending on context) such that we have the family of bounds*

$$(2.3) \quad |\partial_\rho^k \partial_{x^{a_1}} \dots \partial_{x^{a_m}} u| \lesssim_{g,k,m,u} \rho^{-k} \zeta, \quad k, m \geq 0,$$

where the  $x^{a_i}$ 's refer to any of the coordinates in  $\varphi_{\rho,t}$  except for  $\rho$ .

When  $\varphi$  is clear from context, we omit it from notation and write  $\mathcal{O}(\zeta)$ .

2.1.2. *Admissible Spacetimes.* We can now define our class of aAdS spacetimes:

**Definition 2.5.** *Let  $g$  be a Lorentzian metric on  $\mathcal{M}$ . We say that  $(\mathcal{M}, g)$  is an admissible aAdS segment iff the following conditions hold:*

- (1) *There exist symmetric covariant 2-tensors  $\dot{g}$ ,  $\bar{g}$ ,  $E$  on  $\mathcal{M}$ , with  $\dot{g}$  and  $\bar{g}$  independent of  $\rho$ , such that  $g$  can be expressed as*

$$(2.4) \quad g = \rho^{-2}[\dot{g} + \rho^2 \bar{g} + \rho^3 E].$$

- (2)  *$\dot{g}$  and  $\bar{g}$  have the forms*

$$(2.5) \quad \dot{g} = d\rho^2 - dt^2 + \gamma, \quad \bar{g} = \varsigma \cdot d\rho^2 + \bar{\mathfrak{g}},$$

where  $\varsigma \in C^\infty(\mathcal{M})$ , where  $\bar{\mathfrak{g}}$  only has components tangent to the level sets of  $\rho$ , and where  $\gamma$  only has components tangent to the level sets of  $(\rho, t)$ .

- (3) *There exists a finite family  $\Xi$  of coordinate systems on  $\mathcal{S}$  covering all of  $\mathcal{S}$ , such that for any  $\varphi \in \Xi$ , the components with respect to the  $\varphi_{\rho,t}$ -coordinates of  $\dot{g}$ ,  $\bar{g}$ ,  $E$ , and the metric dual  $\dot{g}^{-1}$  of  $\dot{g}$  satisfy<sup>8</sup>*

$$(2.6) \quad \dot{g}_{\alpha\beta} = \mathcal{O}_\varphi(1), \quad \dot{g}^{\alpha\beta} = \mathcal{O}_\varphi(1), \quad \bar{g}_{\alpha\beta} = \mathcal{O}_\varphi(1), \quad E_{\alpha\beta} = \mathcal{O}_\varphi(1).$$

We call this  $\Xi$  a bounded family of coordinates on  $(\mathcal{M}, g)$ .

**Definition 2.6.** *In the context of Definition 2.5, we refer to the Lorentzian manifold  $(\mathcal{I}, \dot{\mathfrak{g}})$ , where  $\dot{\mathfrak{g}} := -dt^2 + \gamma$  is the restriction of  $\dot{g}$  to the  $\mathcal{I}$ -tangent directions, as the induced AdS-type boundary. Furthermore, we will occasionally slightly abuse notation and use  $\mathcal{I}$  to refer to this conformal boundary  $\{\rho = 0\}$  of  $(\mathcal{M}, g)$ .*

<sup>8</sup>Since  $\dot{g}$  and  $\bar{g}$  are independent of  $\rho$ , condition (2.6) simply implies that all  $x^a$ -coordinate derivatives of their components are bounded on  $\mathcal{I}$ .

Suppose  $(\mathcal{M}, g)$  is such an admissible aAdS segment, on which  $\Xi$  is a bounded family of coordinates. Then, the conditions (2.4)–(2.6) can be stated less formally in the following coordinate representation: with respect to any  $\varphi \in \Xi$ ,

$$(2.7) \quad \rho^2 g = (1 + \rho^2 \bar{g}_{\rho\rho}) d\rho^2 + (-dt^2 + \dot{g}_{AB} dx^A dx^B + \rho^2 \bar{g}_{ab} dx^a dx^b) \\ + \mathcal{O}_\varphi(\rho^3) \cdot dx^\alpha dx^\beta.$$

Letting  $r := \rho^{-1}$ , then (2.7) takes the more familiar form (2.1).

The spacetimes described in Definition 2.5 resemble those in [7, Definition 2.6], but with the following specific differences:

- (1) We now allow for the boundary metric  $\dot{g}$  to be time-dependent.
- (2) The metric expansion here (see (2.5) and (2.7)) is more general than is prescribed in [7, Definition 2.6], in which  $\bar{g}_{\rho\rho} \equiv -1$ .
- (3) To compensate for the above, we apply a more restrictive prescription of “ $\mathcal{O}_\varphi$ ” error terms: unlike in [7],  $x^a$ -derivatives do not lose powers of  $\rho$ . Although the conditions here are strictly more stringent than before, they are in many important ways more natural.<sup>9</sup>

**Remark.** We note the conditions  $\dot{g}_{tt} \equiv -1$  and  $\dot{g}_{tA} \equiv 0$  in (2.5) cause only a slight loss of generality, as this is essentially a coordinate gauge condition. In particular, this can be forced by defining  $t$  as the affine parameters of a family of normal timelike geodesics emanating from a cross-section of  $\mathcal{I}$ .

What is nontrivially assumed here, though, is that the foliation defined by this special  $t$  remains globally regular on all of  $\mathcal{I}$ .

**Remark.** In fact, the finiteness of  $\Xi$  in Definition 2.5 is not strictly necessary. However, if  $\Xi$  is to be infinite, then we must also assume that the constants associated with all the “ $\mathcal{O}_\varphi(1)$ ’s” in (2.6) are independent of  $\varphi$ . For clarity and simplicity, we assume finite  $\Xi$ , since this is satisfied by all aAdS spacetimes of interest.

Finally, we fix the following notations:

**Definition 2.7.** Let  $\nabla$  denote the Levi-Civita connection associated with  $g$ , and let  $\nabla$  denote the induced connections on the level sets of  $(\rho, t)$ , i.e., the copies of  $\mathcal{S}$ .

**2.1.3. Fefferman–Graham Spacetimes.** By applying an appropriate change of coordinates, we can convert an admissible aAdS segment into “Fefferman–Graham form”, for which all the information in  $g$  resides in the  $\mathcal{I}$ -tangent directions; see [2]. Our analysis therefore reduces to admissible Fefferman–Graham-aAdS segments, which we define as follows:

**Definition 2.8.** We say that  $(\mathcal{M}, g)$  is an admissible Fefferman–Graham-aAdS (FG-aAdS) segment iff the following conditions hold:

- (1)  $(\mathcal{M}, g)$  is an admissible aAdS segment, as in Definition 2.5.
- (2) Both  $\bar{g}$  and  $E$  only contain components tangent to the level sets of  $\rho$ .

Moreover, for such an admissible FG-aAdS segment  $(\mathcal{M}, g)$ , we let  $\dot{g}$  and  $\bar{g}$  denote the restrictions of  $\dot{g}$  and  $\bar{g}$ , respectively, to the  $\mathcal{I}$ -tangent directions.

The main content of Definition 2.8 is the less formal but more intuitive coordinate expansion (2.2). Indeed, from (2.7) and Definition 2.8, we observe for an

<sup>9</sup>For instance, the asymptotics here are necessary to convert to the Fefferman–Graham gauge.



admissible FG-aAdS segment  $(\mathcal{M}, g)$ : for any  $\varphi \in \Xi$ , where  $\Xi$  is a bounded family of coordinates on  $(\mathcal{M}, g)$ , one has the expansion (2.2) for  $g$ . Through most of this paper, we will work directly with the representation (2.2), with the implicit understanding that the precise descriptions are as in Definitions 2.5 and 2.8.

As already mentioned, in Appendix A, we construct the coordinate transformation mapping a general admissible aAdS segment of Definition 2.5 to an FG-aAdS segment of Definition 2.8.

Finally, we recall that for FG-aAdS segments that also satisfy the Einstein-vacuum equations (1.2), we have the added advantage that  $\bar{\mathfrak{g}}$  is determined by the geometry of  $(\mathcal{I}; \mathring{\mathfrak{g}})$  whenever  $n \geq 3$ . More specifically,  $-\bar{\mathfrak{g}}$  is precisely the Schouten tensor associated with  $\mathring{\mathfrak{g}}$ ; see Appendix B for further discussions.

For example, by a change of the  $\rho$ -variable, one can express AdS spacetime as an FG-aAdS segment. In terms of our current notations, we then have

$$(2.8) \quad \mathring{\mathfrak{g}} = -dt^2 + \tilde{\gamma}, \quad \bar{\mathfrak{g}} = -\frac{1}{2}(dt^2 + \tilde{\gamma}),$$

where  $\tilde{\gamma}$  denotes the canonical metric on the unit sphere  $\mathbb{S}^{n-1}$ . More generally, by employing an appropriate change of variables [5], one can show that any Kerr-AdS spacetime with  $n \geq 3$  also has the expansion (2.8).

**2.2. Horizontal Tensor Fields.** Assume, as detailed in Definition 2.8, an admissible FG-aAdS spacetime segment  $(\mathcal{M}, g)$ . The other half of the formalism we will require in this article is the notion of horizontal and mixed tensor fields on  $\mathcal{M}$ . By “horizontal fields”, we refer to fields on  $\mathcal{M}$  which are tensor fields on each level set of  $(\rho, t)$ , i.e., each copy of  $\mathcal{S}$ . By “mixed fields”, we refer to fields which are combinations of standard and horizontal tensor fields.

We will adopt the same definitions and notations as [7]. We briefly review these below; for more details, see [7, Sect. 2.4].

**2.2.1. Horizontal and Mixed Fields.** The main objects of interest are as follows:

- We denote by  $T_\lambda^\mu \mathcal{M}$  the usual  $(\mu, \lambda)$ -tensor bundle over  $\mathcal{M}$ , consisting of all tensors at all points of  $\mathcal{M}$  of rank  $(\mu, \lambda)$ . The space of smooth sections of  $T_\lambda^\mu \mathcal{M}$ —the *tensor fields* of rank  $(\mu, \lambda)$ —are denoted  $\Gamma T_\lambda^\mu \mathcal{M}$ .
- We denote by  $\underline{T}_l^m \mathcal{M}$  the  $(\mathcal{S})$ -horizontal bundle over  $\mathcal{M}$ , containing all tensors of rank  $(m, l)$  on each level set of  $(\rho, t)$  in  $\mathcal{M}$  (i.e., all *horizontal tensors*). We let  $\Gamma \underline{T}_l^m \mathcal{M}$  denote the space of smooth sections of  $\underline{T}_l^m \mathcal{M}$ , i.e., the *horizontal tensor fields* of rank  $(m, l)$ .
- We generalize and unify the above by defining the *mixed bundles* as

$$(2.9) \quad T_\lambda^\mu \underline{T}_l^m \mathcal{M} := T_\lambda^\mu \mathcal{M} \otimes \underline{T}_l^m \mathcal{M}.$$

Similarly, we let  $\Gamma T_\lambda^\mu \underline{T}_l^m \mathcal{M}$  denote the corresponding space of smooth sections of  $T_\lambda^\mu \underline{T}_l^m \mathcal{M}$ , i.e., the *mixed tensor fields*.

Recall that by duality, we can consider any  $A \in \Gamma T_\lambda^\mu \underline{T}_l^m \mathcal{M}$  as a  $C^\infty(\mathcal{M})$ -multilinear map on the appropriate number of standard and horizontal vector fields and 1-forms. Moreover, note in particular that  $\Gamma T_0^0 \mathcal{M} = \Gamma \underline{T}_0^0 \mathcal{M} = C^\infty(\mathcal{M})$ .

**2.2.2. Connection and Curvature.** Next, recall the Levi-Civita connection  $\nabla$  on  $(\mathcal{M}, g)$  induces a bundle connection  $\nabla$  on any  $T_\lambda^\mu \mathcal{M}$ . These connections in turn induce *horizontal connections*  $\nabla$  on the  $\underline{T}_l^m \mathcal{M}$  by projecting to the level sets of

$(\rho, t)$ . The connections  $\nabla$  and  $\nabla^\sharp$  can then be canonically combined to obtain *mixed connections*—also denoted  $\nabla$  here—on the  $T_\lambda^\mu \underline{T}_l^m \mathcal{M}$ 's.

For practical purposes, the main properties of mixed connections are as follows:

- $\nabla$  annihilates both  $g$  and the restrictions  $\gamma$  of  $g$  to the level sets of  $(\rho, t)$  (both of which can be considered as mixed tensor fields).
- Given a fully covariant mixed tensor field  $A \in \Gamma T_\lambda^0 \underline{T}_l^0 \mathcal{M}$  and a vector field  $X \in \Gamma T_0^1 \mathcal{M}$ , then  $\nabla_X A$  can be characterized by its actions on vector fields: if  $Z_1, \dots, Z_\lambda \in \Gamma T_0^1 \mathcal{M}$ , and if  $Y_1, \dots, Y_l \in \Gamma \underline{T}^1 \mathcal{M}$ , then

$$\begin{aligned} \nabla_X A(Z_1, \dots, Z_\lambda; Y_1, \dots, Y_l) &= X[A(Z_1, \dots, Z_\lambda; Y_1, \dots, Y_l)] \\ &\quad - A(\nabla_X Z_1, \dots, Z_\lambda; Y_1, \dots, Y_l) - \dots \\ &\quad - A(Z_1, \dots, \nabla_X Z_\lambda; Y_1, \dots, Y_l) \\ &\quad - A(Z_1, \dots, Z_\lambda; \nabla_X Y_1, \dots, Y_l) - \dots \\ &\quad - A(Z_1, \dots, Z_\lambda; Y_1, \dots, \nabla_X Y_l). \end{aligned} \tag{2.10}$$

For more details on the basic definitions, see [7, Sect. 2.4.1].

Finally, given any  $A \in \Gamma T_\lambda^\mu \underline{T}_l^m \mathcal{M}$ :

- We define its *mixed covariant differential*  $\nabla A \in \Gamma T_{\lambda+1}^\mu \underline{T}_l^m \mathcal{M}$  to be the mixed tensor field mapping a vector field  $X$  to  $\nabla_X A$ .
- In particular, we can make sense of  $\square A \in \Gamma T_\lambda^\mu \underline{T}_l^m \mathcal{M}$  as the  $g$ -trace of  $\nabla^2 A$ , with the trace being applied to the two  $\nabla^2$ -components.
- The *mixed curvature* is defined as follows: given  $X, Y \in \Gamma T_0^1 \mathcal{M}$ , we set

$$\mathcal{R}A \in \Gamma T_{\lambda+2}^\mu \underline{T}_l^m \mathcal{M}, \quad \mathcal{R}_{XY}[A] := \nabla_{XY}^2 A - \nabla_{YX}^2 A. \tag{2.11}$$

From (2.10) and direct computations, we obtain the following identity:

**Proposition 2.9.** *Let  $\phi \in \Gamma \underline{T}_l^0 \mathcal{M}$ . Then, given any spacetime vector fields  $X, Y$  and horizontal vector fields  $Z_1, \dots, Z_l$ , we have:*

$$\begin{aligned} \mathcal{R}_{XY}\phi(Z_1, \dots, Z_l) &= -\phi(\nabla_X(\nabla_Y Z_1) - \nabla_Y(\nabla_X Z_1) - \nabla_{[X,Y]} Z_1, \dots, Z_l) \\ &\quad - \dots \\ &\quad - \phi(Z_1, \dots, \nabla_X(\nabla_Y Z_l) - \nabla_Y(\nabla_X Z_l) - \nabla_{[X,Y]} Z_l). \end{aligned} \tag{2.12}$$

In particular, if both  $X$  and  $Y$  are also horizontal, then (2.12) reduces to the usual Riemann curvature operator on the level sets of  $(\rho, t)$ .

**2.2.3. Index Conventions.** We will use capital Latin letters to denote *horizontal multi-indices*, i.e., zero or more horizontal indices. Repeated indices represent summations over all individual indices.

Furthermore, for horizontal tensors, we let  $|\cdot|$  denote the pointwise tensor norm:

$$|\phi|^2 := \phi_I \phi^I. \tag{2.13}$$

Note that the above notational conventions also cover the purely scalar case, in which all multi-indices can essentially be ignored.

**2.3. Asymptotic Expansions.** Again, we assume an admissible FG-aAdS space-time segment  $(\mathcal{M}, g)$ . In this section, we compute asymptotic expansions associated with various geometric quantities on  $\mathcal{M}$ .

Here, and in the remainder of this paper, we assume a bounded family  $\Xi$  of coordinates on  $(\mathcal{M}, g)$ , as in Definition 2.5.. Throughout, when we write  $\mathcal{O}(\zeta)$  (see Definition 2.4), we will implicitly assume this to be with respect to some  $\varphi \in \Xi$ .

2.3.1. *Metric and Christoffel Symbol Expansions.* First, we list the asymptotics of the metric and its corresponding Christoffel symbols.

**Proposition 2.10.** *With respect to any  $\varphi \in \Xi$ , the following hold:*

- *The components of  $g$  satisfy*

$$(2.14) \quad g_{\rho\rho} = \rho^{-2}, \quad g_{\rho a} = 0, \quad g_{ab} = \rho^{-2} \mathring{g}_{ab} + \bar{g}_{ab} + \mathcal{O}(\rho).$$

*In particular,*

$$(2.15) \quad g_{tt} = -\rho^{-2} + \bar{g}_{tt} + \mathcal{O}(\rho), \quad g_{tA} = \bar{g}_{tA} + \mathcal{O}(\rho).$$

- *The dual of  $g$  satisfies*

$$(2.16) \quad g^{\rho\rho} = \rho^2, \quad g^{\rho a} = 0, \quad g^{ab} = \rho^2 \mathring{g}^{ab} - \rho^4 \mathring{g}^{ac} \mathring{g}^{bd} \bar{g}_{cd} + \mathcal{O}(\rho^5).$$

*In particular,*

$$(2.17) \quad g^{tt} = -\rho^2 - \rho^4 \bar{g}_{tt} + \mathcal{O}(\rho^5), \quad g^{tA} = \rho^4 \mathring{g}^{AB} \bar{g}_{tB} + \mathcal{O}(\rho^5).$$

- *The Christoffel symbols with respect to these coordinates satisfy*

$$(2.18) \quad \begin{aligned} \Gamma_{\rho\rho}^\rho &= -\rho^{-1}, & \Gamma_{\rho a}^\rho &= 0, \\ \Gamma_{ab}^\rho &= \rho^{-1} \mathring{g}_{ab} + \mathcal{O}(\rho^2), & \Gamma_{\rho\rho}^a &= 0, \\ \Gamma_{\rho b}^a &= -\rho^{-1} \delta_b^a + \rho \mathring{g}^{ac} \bar{g}_{cb} + \mathcal{O}(\rho^2), & \Gamma_{bc}^a &= \mathring{\Gamma}_{bc}^a + \mathcal{O}(\rho^2), \end{aligned}$$

where  $\mathring{\Gamma}_{bc}^a$  denotes the corresponding Christoffel symbol associated with  $\mathring{g}$ . In addition, when  $\Gamma_{bc}^a$  contains a  $t$ -component, we have:<sup>10</sup>

$$(2.19) \quad \Gamma_{ab}^t = \frac{1}{2} \partial_t \mathring{g}_{ab} + \mathcal{O}(\rho^2), \quad \Gamma_{tb}^a = \frac{1}{2} \mathring{g}^{ac} \partial_t \mathring{g}_{cb} + \mathcal{O}(\rho^2).$$

2.3.2. *Curvature Coefficients.* We will also need to compute the asymptotics for the mixed curvature operator  $\mathcal{R}$  defined in (2.11).

**Proposition 2.11.** *Let  $\phi \in \Gamma \underline{T}_l^0 \mathcal{M}$ . Then, with respect to any  $\varphi \in \Xi$ , we have*

$$(2.20) \quad |\mathcal{R}_{\rho a} \phi| \lesssim_{g,l} \rho |\phi|, \quad |\mathcal{R}_{ab} \phi| \lesssim_{g,l} |\phi|.$$

*Proof.* The computations are analogous to those found in [7]; however, since (2.20) contains some nonstandard definitions involving mixed tensor fields, we give some details for the reader's convenience.

First, using that  $\nabla_\alpha \partial_A$  is the orthogonal projection of  $\nabla_\alpha \partial_A$  to the  $(\rho, t)$ -level sets, along with the asymptotic identities in Proposition 2.10, we obtain

$$(2.21) \quad \nabla_\alpha \partial_A = \Gamma_{\alpha A}^B \partial_B + \sum_{B=1}^{n-1} \mathcal{O}(\rho^2) \cdot \partial_B.$$

Differentiating (2.21) and then applying (2.21) yields the identity

$$(2.22) \quad \begin{aligned} \nabla_\alpha (\nabla_\beta \partial_A) - \nabla_\beta (\nabla_\alpha \partial_A) &= \partial_\alpha \Gamma_{\beta A}^B \partial_B - \partial_\beta \Gamma_{\alpha A}^B \partial_B + \Gamma_{\beta A}^B \Gamma_{\alpha B}^C \partial_C \\ &\quad - \Gamma_{\alpha A}^B \Gamma_{\beta B}^C \partial_C + \sum_{C=1}^{n-1} \mathcal{O}(\rho) \cdot \partial_C. \end{aligned}$$

Since (2.18) implies

$$(2.23) \quad \partial_\rho \Gamma_{aA}^B - \partial_a \Gamma_{\rho A}^B = \mathcal{O}(\rho), \quad \Gamma_{aA}^B \Gamma_{\rho B}^C - \Gamma_{\rho A}^B \Gamma_{aB}^C = \mathcal{O}(\rho),$$

<sup>10</sup>The presence of the (leading-order) quantity  $\partial_t \mathring{g}$  in (2.19) is a fundamental difference between the current setting and that of [7].

then combining (2.21) and (2.23) yields the first part of (2.20). Similarly, since

$$(2.24) \quad \partial_a \Gamma_{bA}^B - \partial_b \Gamma_{aA}^B = \mathcal{O}(1), \quad \Gamma_{bA}^B \Gamma_{aB}^C - \Gamma_{aA}^B \Gamma_{bB}^C = \mathcal{O}(1),$$

by (2.18), then (2.21) and (2.24) implies the second part of (2.20).  $\square$

### 3. THE SPACETIME FOLIATION

In this section,  $(\mathcal{M}, g)$  denotes an admissible FG-aAdS segment, see Definitions 2.1 and 2.8. Moreover, for convenience, we normalize the time interval as

$$(3.1) \quad 0 = T_- < t < T_+ = \pi T.$$

**3.1. Construction of the Level Sets.** Analogous to [7], we construct functions whose level sets will be shown to be pseudoconvex near the conformal boundary  $\mathcal{I}$ . For this purpose, we define the following:

**Definition 3.1.** Fix a constant  $\xi \geq 0$ . We then define  $f := f_{T,\xi} : \mathcal{M} \rightarrow \mathbb{R}$  by

$$(3.2) \quad f(\rho, t, x^A) = \frac{\rho}{\eta(t)},$$

where  $\eta : [0, \pi T] \rightarrow \mathbb{R}$  satisfies:

$$(3.3) \quad \eta(t) := \begin{cases} \exp\left(\frac{\xi}{4}t\right) \sin(\mu t) & t \in [0, \frac{\pi T}{2}), \\ \exp\left(\frac{\xi}{4}(\pi T - t)\right) \sin(\mu(\pi T - t)) & t \in [\frac{\pi T}{2}, \pi T]. \end{cases}$$

Here,  $\mu$  is the unique constant satisfying

$$(3.4) \quad T = \frac{2}{\mu} - \frac{2}{\mu\pi} \arctan\left(\frac{4\mu}{\xi}\right), \quad \frac{1}{T} \leq \mu < \frac{2}{T}.$$

Note that the level sets of  $f$  foliate a neighborhood of  $\mathcal{I}$  in  $\mathcal{M}$ . This specific choice of  $\eta$  will be justified in the developments below. For now, observe:

- $\eta \in C^2[0, \pi T]$ , and  $\eta$  is smooth on  $[0, \frac{\pi T}{2})$  and  $(\frac{\pi T}{2}, \pi T]$ .
- $\eta$  is strictly positive on  $(0, \pi T)$ , and  $\eta(0) = \eta(\pi T) = 0$ .
- $\eta$  and its derivatives are uniformly bounded. In other words, we have  $\eta = \mathcal{O}(1)$  on each of the intervals  $(0, \frac{\pi T}{2})$  and  $(\frac{\pi T}{2}, \pi T)$ .

**Remark.** Compared to the  $f$  employed in [7], the new element here is the parameter  $\xi$ , which will be used to compensate for the non-static boundary. By choosing  $\xi = 0$ , we recover precisely the corresponding function  $f$  used in [7].<sup>11</sup>

A technical issue here that was not encountered in [7] is that  $f$  fails to be smooth. Thus, we often restrict attention to regions in which all objects are smooth:

**Definition 3.2.** We define the following regions,

$$(3.5) \quad \mathcal{M}_- := \mathcal{M} \cap \left\{ t < \frac{\pi T}{2} \right\}, \quad \mathcal{M}_+ := \mathcal{M} \cap \left\{ t > \frac{\pi T}{2} \right\},$$

Furthermore, similar to [7], in our main Carleman estimate, it will often be convenient to work not with  $\nabla^\sharp f$ , but with the following:

**Definition 3.3.** Let  $S$  denote the following rescaling of  $\nabla^\sharp f$ :

$$(3.6) \quad S := f^{n-3} \nabla^\sharp f.$$

<sup>11</sup>In this case,  $\eta(t) = \sin(T^{-1}t)$ , and hence  $f$  is in fact everywhere smooth.

3.1.1. *Asymptotic Expansions.* The next step is to compute asymptotic properties for  $f$ . For this, it will be convenient to introduce a weaker notion (than the  $\mathcal{O}$  of Definition 2.4) of asymptotic error terms, i.e., the notion of asymptotics errors used throughout our previous paper [7]:

**Definition 3.4.** Let  $\zeta \in C(\mathcal{M})$ . We use  $\mathcal{O}_0(\zeta)$  to denote any function  $u$  on an appropriate open subset of  $\mathcal{M}_+ \cup \mathcal{M}_-$  such that we have the family of bounds

$$(3.7) \quad |\partial_{x^{\alpha_1}} \dots \partial_{x^{\alpha_m}} u| \lesssim_{g,m,u} \rho^{-m} \zeta, \quad m \geq 0,$$

where the  $x^{\alpha_i}$ 's refer to any of the (spacetime) coordinates used in Definition 2.8.

**Remark.** The main reason for introducing the above is that while  $f \neq \mathcal{O}(f)$ , we have  $f = \mathcal{O}_0(f)$ , which allows for easier bookkeeping of error terms.

The subsequent proposition lists some basic asymptotic properties of  $f$ :

**Proposition 3.5.** Let  $f, \eta$  be as in Definition 3.1. Then, the gradient of  $f$  satisfies

$$(3.8) \quad \begin{aligned} \nabla^\sharp f &= f\rho\partial_\rho + \eta' f^2[\rho + \mathcal{O}(\rho^3)]\partial_t + \eta' f^2 \cdot \sum_{A=1}^{n-1} \mathcal{O}(\rho^3) \cdot \partial_{x^A}, \\ \nabla^\alpha f \nabla_\alpha f &= f^2[1 - (\eta')^2 f^2 + (\eta')^2 f^2 \cdot \mathcal{O}(\rho^2)] \\ &= f^2 + \mathcal{O}_0(f^4). \end{aligned}$$

In addition,  $\square f$  satisfies

$$(3.9) \quad \square f = -(n-1)f + \mathcal{O}_0(f^3).$$

*Proof.* The first step is to compute derivatives of  $f$ :

$$(3.10) \quad \partial_\rho f = f\rho^{-1}, \quad \partial_t f = -\eta' f^2 \rho^{-1},$$

$$(3.11) \quad \partial_{\rho\rho}^2 f = 0, \quad \partial_{\rho t}^2 f = -\eta' f^2 \rho^{-2}, \quad \partial_{tt}^2 f = -f^2 \rho^{-2}[\eta'' \rho - 2(\eta')^2 f].$$

Both equations in (3.8) follow from (2.16), (2.17), and (3.10). Next, from (2.18), (2.19), and (3.11), we obtain expansions for components of  $\nabla^2 f$ :

$$(3.12) \quad \begin{aligned} \nabla_{\rho\rho} f &= f\rho^{-2}, \\ \nabla_{\rho t} f &= -\eta' f^2 \rho^{-2}[2 + \rho^2 \bar{g}_{tt} + \mathcal{O}(\rho^3)], \\ \nabla_{tt} f &= f\rho^{-2}[1 - \eta'' f \rho + 2(\eta')^2 f^2 + \mathcal{O}(\rho^3) + \eta' f \cdot \mathcal{O}(\rho^3)], \\ \nabla_{AB} f &= -f\rho^{-2}[\bar{g}_{AB} + \mathcal{O}(\rho^3)] + \frac{1}{2}\eta' f^2 \rho^{-2}[\rho \partial_t \bar{g}_{AB} + \mathcal{O}(\rho^3)], \\ \nabla_{\rho A} f &= -\eta' f^2 \rho^{-2}[\rho^2 \bar{g}_{tA} + \mathcal{O}(\rho^3)], \\ \nabla_{tA} f &= f\rho^{-2} \cdot \mathcal{O}(\rho^3) + \eta' f^2 \rho^{-2} \cdot \mathcal{O}(\rho^3). \end{aligned}$$

The final identity (3.9) now follows from (2.16), (2.17), and (3.12).  $\square$

**Corollary 3.6.**  $S$  satisfies the following asymptotic properties:

$$(3.13) \quad \nabla^\alpha S_\alpha = -2f^{n-2} + \mathcal{O}_0(f^n), \quad S^\alpha S_\alpha = f^{2n-4} + \mathcal{O}_0(f^{2n-2}).$$

As in [7], the  $\mathcal{O}_0$ -classes satisfy systematic derivative properties:

**Proposition 3.7.** Let  $\zeta \in C(\mathcal{M})$ , and suppose  $u = \mathcal{O}_0(\zeta)$  is smooth. Then,

$$(3.14) \quad \square u = \mathcal{O}_0(\zeta), \quad \nabla^\alpha f \nabla_\alpha u = \mathcal{O}_0(f\zeta).$$

*Proof.* These are consequences of Proposition 2.10 and (3.8).  $\square$

3.1.2. *Adapted frames.* Similar to [7], we define a collection of orthonormal frames adapted to the foliation by the level sets of  $f$ :

**Definition 3.8.** *We define local frames  $(N, V, E_1, \dots, E_{n-1})$  as follows:<sup>12</sup>*

- *Let  $(E_1, \dots, E_{n-1})$  denote local orthonormal frames on the level sets of  $(\rho, t)$ . Note that by (2.14), these frames can be chosen such that*

$$(3.15) \quad E_X := \rho E_X^A \partial_A, \quad E_X^A = \mathcal{O}(1).$$

- *Let  $N$  denote the inward-pointing unit normal to level sets of  $f$ :*

$$(3.16) \quad N := |\nabla^\alpha f \nabla_\alpha f|^{-\frac{1}{2}} \nabla^\sharp f.$$

- *The final (future, timelike) frame component is then given by:*

$$(3.17) \quad V := |g(\tilde{V}, \tilde{V})|^{-\frac{1}{2}} \tilde{V},$$

$$\tilde{V} := \partial_t + \eta' f \partial_\rho - \sum_{X=1}^{n-1} g(\partial_t + \eta' f \partial_\rho, E_X) \cdot E_X.$$

Direct computations using (2.14) and (3.8) then yield the following:

**Proposition 3.9.** *The frames  $(N, V, E_X)$  in (3.15)-(3.17) are orthonormal. Also:*

- *$N$  and  $V$  have asymptotic expansions*

$$(3.18) \quad N = [1 - (\eta')^2 f^2 + \mathcal{O}_0(f^2 \rho^2)]^{-\frac{1}{2}} \left[ \rho \partial_\rho + \eta' f \rho \partial_t + \sum_a \mathcal{O}_0(f \rho^3) \cdot \partial_a \right],$$

$$V = [1 - (\eta')^2 f^2 - \bar{g}_{tt} \rho^2 + \mathcal{O}_0(\rho^3)]^{-\frac{1}{2}} \cdot \left\{ \rho \partial_t + \eta' f \rho \partial_\rho - \rho^3 E_X^A E_X^B \bar{g}_{tA} \partial_B + \sum_B \mathcal{O}_0(\rho^4) \cdot \partial_B \right\}.$$

- *Furthermore, for  $f \ll_g 1$ , the following inversion formulas hold:*

$$(3.19) \quad [1 - (\eta')^2 f^2]^{\frac{1}{2}} \rho \partial_\rho = [1 + \mathcal{O}_0(\rho^2)] N - [\eta' f + \mathcal{O}_0(\rho^2)] V + \sum_{X=1}^{n-1} \mathcal{O}_0(\rho^2) \cdot E_X,$$

$$[1 - (\eta')^2 f^2]^{\frac{1}{2}} \rho \partial_t = [1 + \mathcal{O}_0(\rho^2)] V - [\eta' f + \mathcal{O}_0(\rho^2)] N + \sum_{X=1}^{n-1} \mathcal{O}_0(\rho^2) \cdot E_X.$$

We will also require the following curvature bounds involving the above frames:

**Proposition 3.10.** *Let  $\phi \in \Gamma \underline{T}_l^0 \mathcal{M}$ . Then, with  $(N, V, E_X)$  as in (3.15)-(3.17),*

$$(3.20) \quad |\mathcal{R}_{NV} \phi| \lesssim_{g,l} \rho^3 |\phi|, \quad |\mathcal{R}_{NE_X} \phi| \lesssim_{g,l} f \rho^2 |\phi|.$$

*Proof.* This follows from using (2.20) along with (3.15) and (3.18).  $\square$

---

<sup>12</sup> $V$  here corresponds to the vector field “ $T$ ” in [7].

**3.2. Pseudoconvexity.** In order to determine the pseudoconvexity properties of the level sets of  $f$ , we need to compute the components of  $\nabla^2 f$  in the frame (3.15)-(3.17). More precisely, we must compute the frame components of

$$(3.21) \quad Q_{\xi,\zeta} = -\nabla^2 f - w_{\xi,\zeta} \cdot g,$$

for a suitable function  $w_{\xi,\zeta}$ , to be specified below in (3.22). That  $Q_{\xi,\zeta}$  is positive-definite for vectors tangent to the level sets of  $f$  then implies that these hypersurfaces are pseudoconvex; see Definition 2.13 and Proposition 2.14 in [7].

In the context of our Carleman estimates, it will be more convenient to express this positivity in terms of  $\nabla S$  rather than  $\nabla^2 f$ . For this purpose, we define:

**Definition 3.11.** *Given a constant  $\xi \geq 0$  and  $\zeta \in C^\infty(\mathcal{M})$ , we let*

$$(3.22) \quad w_{\xi,\zeta} := f - \frac{1}{2} f \rho^2 \xi \frac{|\eta'|}{\eta} + f \rho^2 \zeta,$$

as well as the following modified deformation tensor,

$$(3.23) \quad \pi_{\xi,\zeta} := -(\nabla S + f^{n-3} w_{\xi,\zeta} \cdot g).$$

Note that  $Q_{\xi,\zeta}$  being positive-definite in directions tangent to the level sets of  $f$  is equivalent to  $\pi_{\xi,\zeta}$  being positive-definite in the same components.

**3.2.1. The Pseudoconvexity Criterion.** We now define our main *pseudoconvexity criterion*, which is *stated only in terms of the metric data at infinity*:

**Definition 3.12.** *We say that the pseudoconvexity property holds at  $\mathcal{I}$  iff there are constants  $K > 0$ ,  $\xi \geq 0$  and a function  $\zeta \in C^\infty(\mathcal{M})$  such that:*

(1)  $\zeta = \mathcal{O}(1)$ .

(2) For any vector field  $Y := Y^A \partial_A$  on  $\mathcal{I}$  that is tangent to  $\mathcal{S}$ , we have

$$(3.24) \quad |\mathcal{L}_{\partial_t} \mathring{\mathfrak{g}}(Y, Y)| \leq \xi \cdot \mathring{\mathfrak{g}}(Y, Y).$$

(3) For any vector field  $X := X^t \partial_t + X^A \partial_A$  on  $\mathcal{I}$ , the tensor field

$$(3.25) \quad \mathbf{Q}_{\xi,\zeta} := -\bar{\mathfrak{g}} - \left( \mu^2 + \frac{\xi^2}{16} \right) dt^2 - \zeta \mathring{\mathfrak{g}},$$

where  $\mu$  is defined implicitly by (3.4), satisfies the positivity property

$$(3.26) \quad \mathbf{Q}_{\xi,\zeta}(X, X) \geq K[(X^t)^2 + \mathring{\mathfrak{g}}_{AB} X^A X^B].$$

The main point of this development is that Definition 3.12, which is a condition purely on the metric asymptotics at infinity  $\mathcal{I}$ , implies that the level sets of  $f$  in the spacetime are indeed pseudoconvex, at least for  $f \ll_g 1$ . This is captured in the form that we will use later through the following theorem:

**Theorem 3.13.** *Suppose the pseudoconvexity property holds at  $\mathcal{I}$ , and let  $K$ ,  $\xi$ ,  $\zeta$  be the parameters from Definition 3.12. Then, for any 1-form  $\theta$  on  $\mathcal{M}$ ,*

$$(3.27) \quad \pi_{\xi,\zeta}^{\alpha\beta} \theta_\alpha \theta_\beta \geq [K f^{n-2} \rho^2 + \mathcal{O}_0(f^{n-1} \rho^2)] \left( |\theta(V)|^2 + \sum_{X=1}^{n-1} |\theta(E_X)|^2 \right) - [(n-1)f^{n-2} + \mathcal{O}_0(f^n)] |\theta(N)|^2.$$

3.2.2. *Proof of Theorem 3.13.* The main step in of the proof is the computation for  $\pi_{\xi,\zeta}$ , which proceeds analogously to that in [7].

**Lemma 3.14.**  $\pi_{\xi,\zeta}$  is symmetric, and its  $f$ -tangent components satisfy

$$(3.28) \quad \begin{aligned} \pi_{\xi,\zeta}(V, V) &= \frac{1}{\eta} \left( \eta'' - \frac{\xi}{2} |\eta'| \right) f^{n-2} \rho^2 - (\bar{\mathfrak{g}}_{tt} + \zeta \bar{\mathfrak{g}}_{tt}^{\circ}) f^{n-2} \rho^2 + \mathcal{O}_0(f^{n-1} \rho^2), \\ \pi_{\xi,\zeta}(V, E_X) &= -E_X^A (\bar{\mathfrak{g}}_{tA} + \zeta \bar{\mathfrak{g}}_{tA}^{\circ}) \cdot f^{n-2} \rho^2 + \mathcal{O}_0(f^{n-1} \rho^2), \\ \pi_{\xi,\zeta}(E_X, E_Y) &= \frac{1}{2\eta} \cdot E_X^A E_Y^B (|\eta'| \xi \bar{\mathfrak{g}}_{AB} - \eta' \partial_t \bar{\mathfrak{g}}_{AB}) f^{n-2} \rho^2 \\ &\quad - E_X^A E_Y^B (\bar{\mathfrak{g}}_{AB} + \zeta \bar{\mathfrak{g}}_{AB}^{\circ}) f^{n-2} \rho^2 + \mathcal{O}_0(f^{n-1} \rho^3). \end{aligned}$$

Moreover, the remaining components of  $\pi_{\xi,\zeta}$  satisfy

$$(3.29) \quad \begin{aligned} \pi_{\xi,\zeta}(N, N) &= -(n-1) f^{n-2} + \mathcal{O}_0(f^n), \\ \pi_{\xi,\zeta}(N, V) &= \mathcal{O}_0(f^n \rho), \\ \pi_{\xi,\zeta}(N, E_X) &= \mathcal{O}_0(f^{n-1} \rho^2). \end{aligned}$$

*Proof.* The main computations behind (3.28) and (3.29) are the formulas for  $\nabla^2 f$ , with respect to the aforementioned orthonormal frames. Similar to [7], we will need more precise expansions for components tangent to the level sets of  $f$ :

$$(3.30) \quad \begin{aligned} \nabla_{VV} f &= f + (\bar{\mathfrak{g}}_{tt} - \eta'' \eta^{-1}) f \rho^2 + \mathcal{O}_0(f^2 \rho^2), \\ \nabla_{VE_X} f &= E_X^A \bar{\mathfrak{g}}_{tA} \cdot f \rho^2 + \mathcal{O}_0(f^2 \rho^2), \\ \nabla_{E_X E_Y} f &= -\delta_{XY} f + E_X^A E_Y^B \left( \frac{1}{2} \eta' \eta^{-1} \partial_t \bar{\mathfrak{g}}_{AB} + \bar{\mathfrak{g}}_{AB} \right) f \rho^2 + \mathcal{O}_0(f \rho^3). \end{aligned}$$

For the remaining components of  $\nabla^2 f$ , we have

$$(3.31) \quad \begin{aligned} \nabla_{NN} f &= f - 2(\eta') 2f^3 + \mathcal{O}_0(f^5) = f + \mathcal{O}_0(f^3), \\ \nabla_{NV} f &= [1 + \mathcal{O}_0(f^2)] [-\eta' \eta'' f^3 \rho + \mathcal{O}_0(f^2 \rho^2)] = \mathcal{O}_0(f^3 \rho), \\ \nabla_{NE_X} f &= \mathcal{O}_0(f^2 \rho^2). \end{aligned}$$

Combining (3.23), (3.30), and (3.31) results in (3.28) and (3.29).  $\square$

**Remark.** We note that  $\pi_{\xi,\zeta}(N, V)$  behaves worse compared to the static case considered in [7], while the other components behave similarly compared to [7].

The following properties of  $\eta$  can be verified through direct computations:

**Lemma 3.15.** Given  $\xi > 0$  and  $T > 0$ , the function  $\eta$  in (3.3) satisfies

$$(3.32) \quad \eta''(t) - \frac{\xi}{2} |\eta'(t)| + \left( \frac{\xi^2}{16} + \mu^2 \right) \eta(t) = 0, \quad t \in \left( 0, \frac{\pi T}{2} \right) \cup \left( \frac{\pi T}{2}, T \right).$$

Furthermore,  $\eta'''$  has jump discontinuity at  $\frac{\pi T}{2}$ , since

$$(3.33) \quad \begin{aligned} \lim_{t \rightarrow \frac{\pi T}{2}^-} \eta''' &= -\frac{\xi}{2} \left( \frac{\xi^2}{16} + \mu^2 \right) \exp \left( \frac{\pi T \xi}{8} \right) \sin \left( \frac{\pi T \mu}{2} \right) < 0, \\ \lim_{t \rightarrow \frac{\pi T}{2}^+} \eta''' &= +\frac{\xi}{2} \left( \frac{\xi^2}{16} + \mu^2 \right) \exp \left( \frac{\pi T \xi}{8} \right) \sin \left( \frac{\pi T \mu}{2} \right) > 0. \end{aligned}$$

By combining Definition 3.12 with (3.28) and (3.32), we connect  $\pi_{\xi,\zeta}$  and  $\mathbf{Q}_{\xi,\zeta}$ :



**Lemma 3.16.** *The following identities hold:*

$$\begin{aligned}
 (3.34) \quad \pi_{\xi,\zeta}(V, V) &= f^{n-2}\rho^2 \cdot \mathbf{Q}_{\xi,\zeta}(\partial_t, \partial_t) + \mathcal{O}_0(f^{n-1}\rho^2), \\
 \pi_{\xi,\zeta}(V, E_X) &= f^{n-2}\rho^2 \cdot \mathbf{Q}_{\xi,\zeta}(\partial_t, E_X^A \partial_A) + \mathcal{O}_0(f^{n-1}\rho^2), \\
 \pi_{\xi,\zeta}(E_X, E_Y) &= \frac{1}{2\eta} f^{n-2}\rho^2 \cdot E_X^A E_Y^B (|\eta'| \xi \mathring{\mathfrak{g}}_{AB} - \eta' \partial_t \mathring{\mathfrak{g}}_{AB}) \\
 &\quad + f^{n-2}\rho^2 \cdot \mathbf{Q}_{\xi,\zeta}(E_X^A \partial_A, E_Y^B \partial_B) + \mathcal{O}_0(f^{n-1}\rho^2).
 \end{aligned}$$

Finally, let  $\theta$  be as in the hypotheses of Theorem 3.13, and define in addition

$$(3.35) \quad \emptyset := \sum_{X=1}^{n-1} \theta(E_X) \cdot E_X^A \partial_A, \quad \check{\theta} := -\theta(V) \cdot \partial_t + \emptyset,$$

which can be viewed as vector fields on  $\mathcal{M}$  or as  $\rho$ -parametrized families of vector fields on  $\mathcal{I}$ . Using (3.29), (3.34), and that  $(N, V, E_X)$  is orthonormal, we have

$$\begin{aligned}
 (3.36) \quad \pi_{\xi,\zeta}^{\alpha\beta} \theta_\alpha \theta_\beta &= f^{n-2}\rho^2 \cdot \mathbf{Q}_{\xi,\zeta}^{ab} \check{\theta}_a \check{\theta}_b + \frac{1}{2\eta} f^{n-2}\rho^2 \cdot (|\eta'| \xi \mathring{\mathfrak{g}}_{AB} - \eta' \partial_t \mathring{\mathfrak{g}}_{AB}) \emptyset^A \emptyset^B \\
 &\quad + \sum_{\mathfrak{A}, \mathfrak{B} \in \{V, E_1, \dots, E_{n-1}\}} \mathcal{O}_0(f^{n-1}\rho^2) \cdot \theta(\mathfrak{A}) \theta(\mathfrak{B}) \\
 &\quad - [(n-1)f^{n-2} + \mathcal{O}_0(f^n)] \cdot |\theta(N)|^2 \\
 &\quad + \sum_{\mathfrak{A} \in \{V, E_1, \dots, E_{n-1}\}} \mathcal{O}_0(f^n \rho) \cdot \theta(N) \theta(\mathfrak{A}) \\
 &= I_1 + I_2 + I_3 + I_4 + I_5.
 \end{aligned}$$

By (3.24), we have

$$(3.37) \quad I_2 \geq 0,$$

while (3.26), along with (2.14) and the identity  $g(E_X, E_Y) = \delta_{XY}$ , implies

$$\begin{aligned}
 (3.38) \quad I_1 &\geq K f^{n-2}\rho^2 \left[ |\theta(V)|^2 + \sum_{X,Y=1}^{n-1} \theta(E_X) \theta(E_Y) \cdot E_X^A E_Y^B \mathring{\mathfrak{g}}_{AB} \right] \\
 &\geq K f^{n-2}\rho^2 \left\{ |\theta(V)|^2 + [1 + \mathcal{O}_0(\rho^2)] \sum_{X=1}^{n-1} |\theta(E_X)|^2 \right\}.
 \end{aligned}$$

Observe that the remaining terms, which are errors, satisfy

$$\begin{aligned}
 (3.39) \quad I_3 &\geq \mathcal{O}_0(f^{n-1}\rho^2) \cdot \left[ |\theta(V)|^2 + \sum_{X=1}^{n-1} |\theta(E_X)|^2 \right], \\
 I_5 &\geq \mathcal{O}_0(f^n) \cdot |\theta(N)|^2 + \mathcal{O}_0(f^n \rho^2) \cdot \left[ |\theta(V)|^2 + \sum_{X=1}^{n-1} |\theta(E_X)|^2 \right].
 \end{aligned}$$

Combining (3.36)-(3.39) yields our desired inequality (3.27).

**3.2.3. Some Examples.** Recall that as an FG-aAdS segment, AdS spacetime (and more generally, the Kerr-AdS family for  $n \geq 3$ , after a change of coordinates from the usual Boyer-Lindquist coordinates, cf. [5]) have expansion (2.8). We now check when the pseudoconvexity property of Definition 3.12 is satisfied.

Since  $\mathring{\mathfrak{g}}$  is static, we can take  $\xi = 0$ . Note that:

- (3.24) is trivially satisfied.
- $\mu = T^{-1}$  from (3.4), hence  $\mathbf{Q}_{0,\zeta}$  from (3.25) is given by

$$(3.40) \quad \mathbf{Q}_{0,\zeta} = \left( \frac{1}{2} - \frac{1}{T} + \zeta \right) dt^2 + \left( \frac{1}{2} - \zeta \right) \tilde{\gamma}.$$

Observe that one can find  $\zeta$  such that (3.40) is positive-definite if and only if  $T > 1$ . In other words, the pseudoconvexity property is satisfied for AdS and Kerr-AdS spacetime if and only if we consider a segment with time length strictly greater than  $\pi$ . This confirms the equivalent results on AdS spacetime established in [7].

Moreover, one can now easily construct a large class of examples satisfying the pseudoconvexity criterion by taking (static or nonstatic) perturbations of  $\mathring{\mathbf{g}}$  and  $\bar{\mathbf{g}}$  from (2.8). In particular, for a small enough perturbation, there would be some  $\varepsilon > 0$  such that Definition 3.12 is satisfied for  $\xi = \varepsilon$  and for  $T > 1 + \varepsilon$ —that is, a time length slightly greater than  $(1 + \varepsilon)\pi$ .

Finally, in any setting for which  $\mathring{\mathbf{g}}$  is static and  $(\mathcal{M}, g)$  is Einstein-vacuum, we can directly relate the pseudoconvexity condition with positive curvature of the level sets of  $t$  on the conformal boundary  $\mathcal{I}$ . See Appendix B for details.

#### 4. THE STATIC BORDERLINE CASE

In this section, we consider a special class of “borderline” static boundary metrics. Let  $(\mathcal{M}, g)$  be an admissible FG-aAdS segment, with

$$(4.1) \quad \mathcal{S} := \mathbb{S}^{n-1}, \quad 0 = T_- < t < T_+ = \pi.$$

We assume static boundary data given by

$$(4.2) \quad \mathring{\mathbf{g}} := -dt^2 + \tilde{\gamma}, \quad \bar{\mathbf{g}} := -\frac{1}{2}(dt^2 + \tilde{\gamma}),$$

i.e., the same boundary data as for AdS spacetime itself.<sup>13</sup>

In particular, *any  $g$  satisfying (4.1) and (4.2) fails the pseudoconvexity criterion of Definition 3.12 for any  $\xi \geq 0$ .* (When  $\xi = 0$ , the pseudoconvexity criterion holds for any  $T_+ > \pi$  but barely fails for  $T_+ = \pi$ .) Because of this, we must try to extract pseudoconvexity at one order higher than  $\bar{\mathbf{g}}$ . Consequently, we assume, in addition to  $(\mathcal{M}, g)$  being an FG-aadS segment, that  $g$  has the refined expansion

$$(4.3) \quad g = \rho^{-2} d\rho^2 + [\rho^{-2} \mathring{\mathbf{g}}_{ab} + \bar{\mathbf{g}}_{ab} + \rho \hat{\mathbf{g}}_{ab} + \mathcal{O}(\rho^2)] dx^a dx^b,$$

i.e., we stipulate an extra third-order term in  $\hat{\mathbf{g}}$  in the expansion for  $g$ .

**Definition 4.1.** *We refer to any FG-aAdS segment  $(\mathcal{M}, g)$  that also satisfies (4.1)-(4.3) as a borderline FG-aAdS segment.*

**Remark.** *The error terms  $\mathcal{O}(\rho^2)$  in (4.3) can be replaced by slightly weaker decay, such as  $\mathcal{O}(\rho^2 \log \rho)$  or  $\mathcal{O}(\rho^{2-\delta})$ . However, to avoid further cluttering the existing presentation, we do not pursue this here.*

**Remark.** *For metrics satisfying (1.2), there is actually no loss in working with the expansion (4.3). If  $g$  satisfies (1.2) and  $n$  is not equal to 2 or 4, then the FG-expansion of  $g$  is precisely given by (4.3); see [2]. On the other hand, if  $n = 2$  or  $n = 4$ , then generally there is also an  $\mathcal{O}(\rho^2 \log \rho)$ -term present in the expansion (4.3). However, in that case, (1.2) and (4.2) imply  $\hat{\mathbf{g}} = 0$ , and we would not be able to extract pseudoconvexity at this level (that is, Definition 4.7 fails to be satisfied).*

<sup>13</sup>Recall that  $\tilde{\gamma}$  denotes the standard metric on the unit sphere  $\mathbb{S}^{n-1}$ .

**4.1. Metric Computations.** Because of the extra term  $\hat{\mathbf{g}}$  in (4.3), we must recompute all the asymptotic expansions obtained in Sections 2 and 3. Since the computations here are similar to their previous counterparts, we only list the main results here and leave details to the reader.

**Proposition 4.2.** *Let  $(\mathcal{M}, g)$  be a borderline FG-aAdS segment.*

- *The components of  $g$  satisfy*

$$(4.4) \quad \begin{aligned} g_{\rho\rho} &= \rho^{-2}, & g_{\rho a} &= 0, \\ g_{tt} &= -\rho^{-2} - \frac{1}{2} + \rho \hat{\mathbf{g}}_{tt} + \mathcal{O}(\rho^2), & g_{tA} &= \rho \hat{\mathbf{g}}_{tA} + \mathcal{O}(\rho^2), \\ g_{AB} &= \rho^{-2} \tilde{\gamma}_{AB} - \frac{1}{2} \tilde{\gamma}_{AB} + \rho \hat{\mathbf{g}}_{AB} + \mathcal{O}(\rho^2). \end{aligned}$$

- *The dual of  $g$  satisfies*

$$(4.5) \quad \begin{aligned} g^{\rho\rho} &= \rho^2, & g^{\rho a} &= 0, \\ g^{tt} &= -\rho^2 + \frac{1}{2} \rho^4 - \rho^5 \hat{\mathbf{g}}_{tt} + \mathcal{O}(\rho^6), & g^{tA} &= \rho^5 \tilde{\gamma}^{AB} \hat{\mathbf{g}}_{tB} + \mathcal{O}(\rho^6), \\ g^{AB} &= \rho^2 \tilde{\gamma}^{AB} + \frac{1}{2} \rho^4 \tilde{\gamma}^{AB} - \rho^5 \tilde{\gamma}^{AC} \tilde{\gamma}^{BD} \hat{\mathbf{g}}_{CD} + \mathcal{O}(\rho^6). \end{aligned}$$

- *The Christoffel symbols with respect to coordinate systems  $\varphi \in \Xi$  satisfy*

$$(4.6) \quad \begin{aligned} \Gamma_{\rho\rho}^\rho &= -\rho^{-1}, & \Gamma_{\rho a}^\rho &= 0, \\ \Gamma_{tt}^\rho &= -\rho^{-1} - \frac{1}{2} \rho^2 \hat{\mathbf{g}}_{tt} + \mathcal{O}(\rho^3), & \Gamma_{tA}^\rho &= -\frac{1}{2} \rho^2 \hat{\mathbf{g}}_{tA} + \mathcal{O}(\rho^3), \\ \Gamma_{AB}^\rho &= \rho^{-1} \tilde{\gamma}_{AB} - \frac{1}{2} \rho^2 \hat{\mathbf{g}}_{AB} + \mathcal{O}(\rho^3), & \Gamma_{\rho\rho}^a &= 0, \\ \Gamma_{\rho t}^t &= -\rho^{-1} + \frac{1}{2} \rho - \frac{3}{2} \rho^2 \hat{\mathbf{g}}_{tt} + \mathcal{O}(\rho^3), & \Gamma_{\rho A}^t &= -\frac{3}{2} \rho^2 \hat{\mathbf{g}}_{tA} + \mathcal{O}(\rho^3), \\ \Gamma_{\rho B}^A &= -\rho^{-1} \delta_B^A - \frac{1}{2} \rho \delta_B^A + \frac{3}{2} \rho^2 \tilde{\gamma}^{AC} \hat{\mathbf{g}}_{CB} + \mathcal{O}(\rho^3), \\ \Gamma_{\rho t}^A &= \frac{3}{2} \rho^2 \tilde{\gamma}^{AB} \hat{\mathbf{g}}_{tB} + \mathcal{O}(\rho^3). \end{aligned}$$

In addition, with  $\mathring{\Gamma}$  denoting the Christoffel symbols for  $\mathring{\mathbf{g}}$ , we have

$$(4.7) \quad \Gamma_{bc}^a = \mathring{\Gamma}_{bc}^a + \mathcal{O}(\rho^3), \quad \Gamma_{ab}^t = \mathcal{O}(\rho^3), \quad \Gamma_{tb}^a = \mathcal{O}(\rho^3).$$

- *With respect coordinate systems  $\varphi \in \Xi$ , we have for any  $\phi \in \Gamma \mathcal{T}_l^0 \mathcal{M}$  that*

$$(4.8) \quad |\mathcal{R}_{\rho a} \phi| \lesssim_{g,l} \rho^2 |\phi|, \quad |\mathcal{R}_{tA} \phi| \lesssim_{g,l} \rho^3 |\phi|, \quad |\mathcal{R}_{AB} \phi| \lesssim_{g,l} |\phi|.$$

**4.2. The Modified Foliation.** We now define the analogues of (3.2) and (3.6):

**Definition 4.3.** *Define the quantities  $\hat{f}$ ,  $\hat{\eta}$  and  $\hat{S}$  by*

$$(4.9) \quad \hat{f} := \hat{\eta}^{-1} \rho \left( 1 - \frac{1}{4} \rho^2 \right), \quad \hat{\eta} := \sin t, \quad \hat{S} := \hat{f}^{n-3} \nabla^\# \hat{f}.$$

The goal once again is to show that the level sets of  $\hat{f}$  are pseudoconvex. However, this pseudoconvexity will now be discerned from  $\hat{\mathbf{g}}$  rather than  $\mathring{\mathbf{g}}$ .

**Remark.** Notice that  $\hat{f}$  contains an extra term  $-\frac{1}{4} \hat{\eta}^{-1} \rho^3$  which has no analogue in (3.2). This is required in order to cancel certain terms so that the leading-order terms in the pseudoconvexity computations are those associated with  $\hat{\mathbf{g}}$ . In

particular, these leading-order terms contain one extra power of  $\rho$  compared to their analogues in Section 3, thus extra care must be taken in order to see them.

**Remark.** In contrast to (3.3), the function  $\hat{\eta}$  here is smooth. Thus, like in [7], here we can avoid difficulties arising from the non-smoothness of  $\eta$  and  $f$ .

**Remark.** Note that  $\rho \lesssim \hat{f}$ , but only when  $\rho \ll 1$ .

**Proposition 4.4.** Let  $(\mathcal{M}, g)$  be an FG-aAdS segment satisfying (4.1)-(4.3). Then, whenever  $\rho \ll_g 1$ , the following asymptotic expansion hold:

- The gradient of  $\hat{f}$  satisfies

$$(4.10) \quad \begin{aligned} \nabla^\sharp \hat{f} &= \hat{f} \left[ \rho - \frac{1}{2} \rho^3 + \mathcal{O}(\rho^5) \right] \partial_\rho + \hat{\eta}' \hat{f}^2 \left[ \rho - \frac{1}{4} \rho^3 + \hat{\mathbf{g}}_{tt} \rho^4 + \mathcal{O}(\rho^5) \right] \partial_t \\ &\quad - \hat{\eta}' \hat{f}^2 [\tilde{\gamma}^{AB} \hat{\mathbf{g}}_{tB} \rho^4 + \mathcal{O}(\rho^5)] \cdot \partial_A, \\ \nabla^\alpha \hat{f} \nabla_\alpha \hat{f} &= \hat{f}^2 [1 - (\hat{\eta}')^2 \hat{f}^2] (1 - \rho^2) - (\hat{\eta}')^2 \hat{f}^4 \rho^3 \hat{\mathbf{g}}_{tt} + \mathcal{O}_0(\hat{f}^4 \rho^4) \\ &= \hat{f}^2 + \mathcal{O}_0(\hat{f}^4). \end{aligned}$$

- The second derivatives of  $\hat{f}$  satisfy:

$$(4.11) \quad \begin{aligned} \nabla_{\rho\rho} \hat{f} &= \hat{f} \rho^{-2} [1 - 2\rho^2 + \mathcal{O}(\rho^4)], \\ \nabla_{\rho t} \hat{f} &= -\hat{\eta}' \hat{f}^2 \rho^{-2} \left[ 2 - \frac{1}{2} \rho^2 + \frac{3}{2} \rho^3 \hat{\mathbf{g}}_{tt} + \mathcal{O}(\rho^4) \right], \\ \nabla_{tt} \hat{f} &= \hat{f} \rho^{-2} \left\{ 1 + 2(\hat{\eta}')^2 \hat{f}^2 + \left[ \frac{1}{2} + (\hat{\eta}')^2 \hat{f}^2 \right] \rho^2 + \frac{1}{2} \hat{\mathbf{g}}_{tt} \rho^3 + \mathcal{O}_0(\hat{f} \rho^3) \right\}, \\ \nabla_{AB} \hat{f} &= -\hat{f} \rho^{-2} \left[ \tilde{\gamma}_{AB} - \frac{1}{2} \rho^2 \tilde{\gamma}_{AB} - \frac{1}{2} \rho^3 \hat{\mathbf{g}}_{AB} + \mathcal{O}_0(\rho^4) \right], \\ \nabla_{\rho A} \hat{f} &= -\hat{\eta}' \hat{f}^2 \rho^{-2} \left[ \frac{3}{2} \rho^3 \hat{\mathbf{g}}_{tA} + \mathcal{O}(\rho^4) \right], \\ \nabla_{tA} \hat{f} &= \hat{f} \rho^{-2} \left[ \frac{1}{2} \rho^3 \hat{\mathbf{g}}_{tA} + \mathcal{O}_0(\hat{f} \rho^4) \right]. \end{aligned}$$

- Furthermore,  $\hat{f}$  satisfies

$$(4.12) \quad \square \hat{f} = -(n-1) \hat{f} + \mathcal{O}_0(\hat{f}^3).$$

4.2.1. *Adapted Frames.* We now define the corresponding frames adapted to level sets of  $\hat{f}$ , and we subsequently list their key properties:

**Definition 4.5.** We define  $(\hat{N}, \hat{V}, \hat{E}_1, \dots, \hat{E}_{n-1})$  as follows:

- Let  $(\hat{E}_1, \dots, \hat{E}_n)$  denote local orthonormal frames on level sets of  $(\rho, t)$ :

$$(4.13) \quad \hat{E}_X := \rho \hat{E}_X^A \partial_A, \quad \hat{E}_X^A = \mathcal{O}(1).$$

- Let  $\hat{N}$  be the inward-pointing unit normal to level sets of  $\hat{f}$ :

$$(4.14) \quad \hat{N} := |\nabla^\alpha \hat{f} \nabla_\alpha \hat{f}|^{-\frac{1}{2}} \nabla^\sharp \hat{f}.$$

- Let  $\hat{V}$  denote the remaining (future, timelike) component:

$$(4.15) \quad \hat{V} := |g(\check{V}, \check{V})|^{-\frac{1}{2}} \check{V},$$

$$\tilde{V} := \left(1 - \frac{3}{4}\rho^2\right) \partial_t + \hat{\eta}' \hat{f} \partial_\rho - \sum_{X=1}^{n-1} g \left( \left(1 - \frac{3}{4}\rho^2\right) \partial_t + \hat{\eta}' \hat{f} \partial_\rho, \hat{E}_X \right) \hat{E}_X.$$

**Proposition 4.6.** *Let  $(\mathcal{M}, g)$  be a borderline FG-aAdS segment. Then, the frames  $(\hat{N}, \hat{V}, \hat{E}_X)$  in (4.13)-(4.15) are orthonormal. Furthermore, whenever  $f, \rho \ll_g 1$ :*

- $\hat{N}$  and  $\hat{V}$  have asymptotic expansions

$$(4.16) \quad \begin{aligned} \hat{N} &= [1 - (\hat{\eta}')^2 \hat{f}^2]^{-\frac{1}{2}} [\rho + \mathcal{O}_0(\hat{f} \rho^4)] \partial_\rho \\ &\quad + [1 - (\hat{\eta}')^2 \hat{f}^2]^{-\frac{1}{2}} \hat{\eta}' \hat{f} \left[ \rho + \frac{1}{4} \rho^3 + \hat{\mathbf{g}}_{tt} \rho^4 + \mathcal{O}_0(\hat{f} \rho^4) \right] \partial_t \\ &\quad + [1 - (\hat{\eta}')^2 \hat{f}^2]^{-\frac{1}{2}} \hat{\eta}' \hat{f} [\tilde{\gamma}^{AB} \hat{\mathbf{g}}_{tB} \rho^4 + \mathcal{O}_0(\rho^5)] \partial_A, \\ \hat{V} &= [1 - (\hat{\eta}')^2 \hat{f}^2 - \rho^2 - \hat{\mathbf{g}}_{tt} \rho^3 + \mathcal{O}(\rho^4)]^{-\frac{1}{2}} \\ &\quad \cdot \left[ \left( \rho - \frac{3}{4} \rho^3 \right) \partial_t + \hat{\eta}' \hat{f} \rho \partial_\rho - \rho^4 E_X^A E_X^B \hat{\mathbf{g}}_{tA} \partial_B + \mathcal{O}(\rho^5) \cdot \partial_B \right]. \end{aligned}$$

- The following inversion formulas hold:

$$(4.17) \quad \begin{aligned} [1 - (\hat{\eta}')^2 \hat{f}^2]^{\frac{1}{2}} \rho \partial_\rho &= [1 + \mathcal{O}_0(\rho^2)] \hat{N} - [\hat{\eta}' \hat{f} + \mathcal{O}_0(\rho^2)] \hat{V} + \sum_{X=1}^{n-1} \mathcal{O}_0(\rho^2) \cdot \hat{E}_X, \\ [1 - (\hat{\eta}')^2 \hat{f}^2]^{\frac{1}{2}} \rho \partial_t &= [1 + \mathcal{O}_0(\rho^2)] \hat{V} - [\hat{\eta}' \hat{f} + \mathcal{O}_0(\rho^2)] \hat{N} + \sum_{X=1}^{n-1} \mathcal{O}_0(\rho^2) \cdot \hat{E}_X. \end{aligned}$$

- For any  $\phi \in \Gamma \underline{T}_l^0 \mathcal{M}$ , we have that

$$(4.18) \quad |\mathcal{R}_{\hat{N}\hat{V}} \phi| \lesssim_{g,l} \rho^4 |\phi|, \quad |\mathcal{R}_{\hat{N}\hat{E}_X} \phi| \lesssim_{g,l} \rho^4 |\phi|.$$

**4.3. Pseudoconvexity.** We now define the analogue of the pseudoconvexity property in Definition 3.12 in our current borderline case. Note that this borderline setting is specially constructed so that Definition 3.12 barely fails to apply.<sup>14</sup> Thus, our refined pseudoconvexity condition below looks for positivity at one order higher—indeed, we will show pseudoconvexity can be extracted precisely from the positivity of the new term  $\hat{\mathbf{g}}$  in the metric expansion (4.3).

**Definition 4.7.** *We say that the borderline pseudoconvexity property holds on  $\mathcal{I}$  iff there exists  $K > 0$  and  $\zeta \in C^\infty(\mathcal{M})$  such that:<sup>15</sup>*

(1)  $\zeta = \mathcal{O}(1)$ .

(2) For any vector field  $X := X^t \partial_t + X^A \partial_A$  on  $\mathcal{I}$ , the tensor field

$$(4.19) \quad \hat{\mathbf{Q}}_\zeta := -\frac{3}{2} \hat{\mathbf{g}} - \zeta \hat{\mathbf{g}},$$

satisfies the positivity property

$$(4.20) \quad \hat{\mathbf{Q}}_\zeta(X, X) \geq K[(X^t)^2 + \hat{\mathbf{g}}_{AB} X^A X^B].$$

As before, we show that the borderline pseudoconvexity property implies that the level sets of  $\hat{f}$  are pseudoconvex, although the pseudoconvexity degenerates at one order higher than in Theorem 3.13 as one goes to  $\mathcal{I}$ .

<sup>14</sup>In particular, at best, one can only find  $\zeta$  such that  $K = 0$  in (3.26).

<sup>15</sup>Since  $\hat{\mathbf{g}}$  is static, we no longer require the parameter  $\xi$  in Definition 3.12.

**Definition 4.8.** Given  $\zeta \in C^\infty(\mathcal{M})$ , we let

$$(4.21) \quad \hat{w}_\zeta := \hat{f} + \hat{f}\rho^3\zeta, \quad \hat{\pi}_{\xi,\zeta} := -(\nabla\hat{S} + \hat{f}^{n-3}\hat{w}_\zeta \cdot g).$$

**Theorem 4.9.** Let  $(\mathcal{M}, g)$  be a borderline FG-aAdS segment, and suppose the borderline pseudoconvexity property holds at  $\mathcal{I}$ , with  $K$  and  $\zeta$  being the parameters from Definition 4.7. Then, for any 1-form  $\theta$  on  $\mathcal{M}$ , we have when  $\rho, f \ll_g 1$  that

$$(4.22) \quad \hat{\pi}_{\xi,\zeta}^{\alpha\beta}\theta_\alpha\theta_\beta \geq [K\hat{f}^{n-2}\rho^3 + \mathcal{O}_0(\hat{f}^{n-1}\rho^3)] \left( |\theta(\hat{V})|^2 + \sum_{X=1}^{n-1} |\theta(\hat{E}_X)|^2 \right) - [(n-1)\hat{f}^{n-2} + \mathcal{O}_0(\hat{f}^n)]|\theta(\hat{N})|^2.$$

*Proof.* The proof proceeds similarly to that of Theorem 3.13. The main step is use (4.11), (4.13), and (4.16) in order to expand  $\nabla^2\hat{f}$  with respect to the orthonormal frames  $(\hat{N}, \hat{V}, \hat{E}_1, \dots, \hat{E}_{n-1})$ . From this, we obtain

$$(4.23) \quad \begin{aligned} \hat{\pi}_\zeta(\hat{V}, \hat{V}) &= -\hat{f}^{n-2}\rho^3 \left( \frac{3}{2}\hat{\mathfrak{g}}_{tt} + \zeta\hat{\mathfrak{g}}_{tt} \right) + \mathcal{O}_0(\hat{f}^{n-1}\rho^3), \\ \hat{\pi}_\zeta(\hat{V}, \hat{E}_X) &= -\hat{f}^{n-2}\rho^3 E_X^A \left( \frac{3}{2}\hat{\mathfrak{g}}_{tA} + \zeta\hat{\mathfrak{g}}_{tA} \right) + \mathcal{O}_0(\hat{f}^{n-1}\rho^3), \\ \hat{\pi}_\zeta(\hat{E}_X, \hat{E}_Y) &= -\hat{f}^{n-2}\rho^3 E_X^A E_Y^B \left( \frac{3}{2}\hat{\mathfrak{g}}_{AB} + \zeta\hat{\mathfrak{g}}_{AB} \right) + \mathcal{O}_0(\hat{f}^{n-2}\rho^4), \\ \hat{\pi}_\zeta(\hat{N}, \hat{N}) &= -(n-1)\hat{f}^{n-2} + \mathcal{O}_0(\hat{f}^n), \\ \hat{\pi}_\zeta(\hat{N}, \hat{V}) &= \mathcal{O}_0(\hat{f}^{n-1}\rho^2), \\ \hat{\pi}_\zeta(\hat{N}, \hat{E}_X) &= \mathcal{O}_0(\hat{f}^{n-1}\rho^3). \end{aligned}$$

An important technical point is the following: while the leading-order terms in the first three equations in (4.23) (which imply pseudoconvexity) are more degenerate than in the non-static case, the cross-terms  $\hat{\pi}_\zeta(\hat{N}, \hat{V})$  and  $\hat{\pi}_\zeta(\hat{N}, \hat{E}_X)$  (which we wish to be error terms) also improved by a power of  $\rho$ .

Defining  $\check{\theta}$  as in (3.35), we compute, using (4.23), that

$$(4.24) \quad \begin{aligned} \hat{\pi}_\zeta^{\alpha\beta}\theta_\alpha\theta_\beta &= \hat{f}^{n-2}\rho^3 \cdot \hat{\mathbf{Q}}_\zeta^{ab}\check{\theta}_a\check{\theta}_b + \sum_{\mathfrak{A}, \mathfrak{B} \in \{\hat{V}, \hat{E}_1, \dots, \hat{E}_{n-1}\}} \mathcal{O}_0(\hat{f}^{n-1}\rho^3) \cdot \theta(\mathfrak{A})\theta(\mathfrak{B}) \\ &\quad - [(n-1)\hat{f}^{n-2} + \mathcal{O}_0(\hat{f}^n)] \cdot |\theta(\hat{N})|^2 \\ &\quad + \sum_{\mathfrak{A} \in \{\hat{V}, \hat{E}_1, \dots, \hat{E}_{n-1}\}} \mathcal{O}_0(\hat{f}^{n-1}\rho^2) \cdot \theta(\hat{N})\theta(\mathfrak{A}). \end{aligned}$$

The term containing  $\hat{\mathbf{Q}}_\zeta$  can be handled in the same manner as its analogue in the proof of Theorem 3.13. Since we can bound for any  $\mathfrak{A} \in \{\hat{V}, \hat{E}_1, \dots, \hat{E}_{n-1}\}$ ,

$$\hat{f}^{n-1}\rho^2 \cdot \theta(\hat{N})\theta(\mathfrak{A}) \lesssim \hat{f}^{n-1}\rho \cdot [\theta(\mathfrak{A})]^2 + \hat{f}^{n-1}\rho^3 \cdot [\theta(\hat{N})]^2,$$

then (4.22) follows immediately from (4.20) and (4.24).  $\square$

**4.3.1. Examples.** As mentioned in Section 1, when  $n = 3$ , Schwarzschild-AdS spacetimes satisfy (4.2) and hence can be considered as borderline FG-aAdS segments. Furthermore, letting  $M \in \mathbb{R}$  denote the mass of the spacetime, we have that

$$(4.25) \quad \hat{\mathfrak{g}} = \frac{2}{3}M(2dt^2 + \tilde{\gamma}).$$

In particular, the borderline pseudoconvexity condition of Definition 4.7 fails to hold in the positive-mass case ( $M > 0$ ), as well as for pure AdS spacetime ( $M = 0$ ). However, the borderline pseudoconvexity condition is satisfied for Schwarzschild-AdS spacetimes with *negative* mass.

## 5. THE MAIN RESULTS

We are now prepared to state and prove our main results. We begin by stating the precise local unique continuation property we wish to establish:

**Definition 5.1.** *Let  $(\mathcal{M}, g)$  be an  $(n+1)$ -dimensional admissible FG-aAdS segment, described in Definitions 2.1 and 2.8, and consider the wave equation*

$$(5.1) \quad \square_g \phi + \sigma \phi = \mathcal{G}(\phi, \nabla \phi), \quad \phi \in \Gamma \underline{T}_l^0 \mathcal{M},$$

where  $l \geq 0$ ,  $\sigma \in \mathbb{R}$ , and  $\mathcal{G} : \Gamma \underline{T}_l^0 \mathcal{M} \times \Gamma T_1^0 \underline{T}_l^0 \mathcal{M} \rightarrow \Gamma \underline{T}_l^0 \mathcal{M}$ .

We say the local unique continuation property holds on  $(\mathcal{M}, g)$  for (5.1) iff given any smooth solution  $\phi$  of (5.1) which satisfies both

(1) the vanishing condition<sup>16</sup>

$$(5.2) \quad 0 = \lim_{\rho' \searrow 0} \int_{\{\rho=\rho'\}} [|\nabla_t(\rho^{-\kappa} \phi)|^2 + |\nabla_\rho(\rho^{-\kappa} \phi)|^2 + |\rho^{-\kappa-1} \phi|^2] d\mathring{g},$$

$$\kappa := \begin{cases} \frac{n-2}{2} + \sqrt{\frac{n^2}{4} - \sigma} & \text{if } \sigma \leq \frac{n^2-1}{4}, \\ \frac{n-1}{2} & \text{if } \sigma > \frac{n^2-1}{4}, \end{cases}$$

(2) and the finiteness condition<sup>17</sup>

$$(5.3) \quad \int_{\mathcal{M}} \rho^{2+p} |\nabla \phi|^2 < \infty, \quad 0 < p \ll 1,$$

then  $\phi$  must vanish in an open neighborhood of the conformal boundary  $\mathcal{I}$ .

Our main unique continuation result is then stated as follows:

**Theorem 5.2.** *Consider an  $(n+1)$ -dimensional admissible FG-aAdS segment*

$$(\mathcal{M}, g), \quad \mathcal{M} = (0, \rho_0) \times (0, T\pi) \times S,$$

and consider on  $(\mathcal{M}, g)$  the wave equation (5.1), for some  $l \geq 0$  and  $\sigma \in \mathbb{R}$ . Furthermore, assume that the following properties hold:

- (1) The pseudoconvexity property (see Definition 3.12) holds on  $\mathcal{I}$ .
- (2) There exist  $p > 0$  and  $C > 0$  such that for any  $\phi \in \Gamma \underline{T}_l^0 \mathcal{M}$ ,

$$(5.4) \quad |\mathcal{G}(\phi, \nabla \phi)|^2 \leq C \rho^p [\rho^4 |\nabla_\rho \phi|^2 + \rho^4 |\nabla_t \phi|^2 + \rho^2 |\nabla \phi|^2 + \rho^{2p} |\phi|^2].$$

Then, the local unique continuation property holds on  $(\mathcal{M}, g)$  for (5.1).

In Section 5.1, we establish the Carleman estimate that is the main step in proving Theorem 5.2. Section 5.2 then applies the Carleman estimate to complete the proof of Theorem 5.2. Finally, in Section 5.3, we discuss the corresponding unique continuation result for the “borderline” case.

<sup>16</sup>Note that  $\kappa = \beta_+ - 1$  when  $\sigma \leq (n^2 - 1)/4$ , where  $\beta_+$  is as in (1.5).

<sup>17</sup>This arises from the fact that no vanishing assumption was imposed for  $\nabla \phi$ .

**5.1. The Carleman Estimate.** In this section, we prove the following Carleman estimate on admissible FG-aAdS segments:

**Theorem 5.3.** *Consider an  $(n+1)$ -dimensional admissible FG-aAdS segment*

$$(\mathcal{M}, g), \quad \mathcal{M} = (0, \rho_0) \times (0, T\pi) \times \mathcal{S},$$

*and suppose the pseudoconvexity property holds on  $\mathcal{I}$ , with associated parameters  $K, \xi, \zeta$ . Fix also an integer  $l \geq 0$ , along with constants  $p, \kappa \in \mathbb{R}$  satisfying*

$$(5.5) \quad 0 < p \ll 1, \quad \kappa \geq \frac{n-1}{2}.$$

*In addition, fix constants  $0 < \rho_0 \ll f_0 \ll_{g,l,p,K} 1$ , and define the region*

$$(5.6) \quad \Omega_{f_0, \rho_0} := \{f < f_0, \rho > \rho_0\}.$$

*Then, there exist constants  $C, \mathcal{C} > 0$ , depending on  $g, p$ , and  $K$ , such that for any  $\sigma \in \mathbb{R}$  and  $\lambda \in [1 + \kappa, \infty)$ , and for any  $\phi \in \Gamma_{\mathcal{I}}^0 \mathcal{M}$  such that both  $\phi$  and  $\nabla \phi$  vanish on  $\{f = f_0\}$ , the following inequality holds:*

$$(5.7) \quad \begin{aligned} & \int_{\Omega_{f_0, \rho_0}} f^{n-2-2\kappa} e^{\frac{-2\lambda f^p}{p}} f^{-p} |(\square + \sigma)\phi|^2 \\ & + C\lambda(\lambda^2 + |\sigma|) \int_{\{\rho=\rho_0\}} [|\nabla_t(\rho^{-\kappa}\phi)|^2 + |\nabla_\rho(\rho^{-\kappa}\phi)|^2 + |\rho^{-\kappa-1}\phi|^2] d\mathfrak{g} \\ & \geq C\lambda \int_{\Omega_{f_0, \rho_0}} f^{n-2-2\kappa} e^{\frac{-2\lambda f^p}{p}} (\rho^4 |\nabla_t \phi|^2 + \rho^4 |\nabla_\rho \phi|^2 + \rho^2 |\nabla \phi|^2) \\ & + \lambda[\kappa^2 - (n-2)\kappa + \sigma - (n-1)] \int_{\Omega_{f_0, \rho_0}} f^{n-2-2\kappa} e^{\frac{-2\lambda f^p}{p}} |\phi|^2 \\ & + C\lambda^3 \int_{\Omega_{f_0, \rho_0}} f^{n-2-2\kappa} e^{\frac{-2\lambda f^p}{p}} f^{2p} |\phi|^2. \end{aligned}$$

In the remainder of this subsection, we prove Theorem 5.3. The proof is mostly analogous to the corresponding proof in [7], with the main difference being that we must account for  $f$  and  $\eta$  not being everywhere smooth.

**5.1.1. Pointwise Estimates.** Analogous to [7], we define the following:

- We first construct the Carleman weight for our estimate:

$$(5.8) \quad F := \kappa \cdot \log f + \lambda p^{-1} f^p, \quad \psi := e^{-F} \phi.$$

Let  $'$  denote differentiation with respect to  $f$ , e.g.,

$$(5.9) \quad F' = \kappa f^{-1} + \lambda f^{-1+p}, \quad F'' = -\kappa f^{-2} - \lambda(1-p)f^{-2+p}.$$

- Recalling  $S$  and  $w_{\xi, \zeta}$  (see (3.6) and (3.22), respectively), we define

$$(5.10) \quad S_{\xi, \zeta} \psi := \nabla_S \psi + h_{\xi, \zeta} \psi, \quad h_{\xi, \zeta} := f^{n-3} w_{\xi, \zeta} + \frac{1}{2} \nabla^\alpha S_\alpha.$$

- We define the conjugated wave operator  $\mathcal{L}$  by

$$(5.11) \quad \mathcal{L} := e^{-F} (\square + \sigma) e^F.$$

- Note that the inward unit normal to the level sets of  $\rho$  is given by

$$(5.12) \quad \mathcal{N} := |\nabla^\alpha \rho \nabla_\alpha \rho|^{-\frac{1}{2}} \nabla^\sharp \rho = \rho \partial_\rho.$$

The key step in proving Theorem 5.3 is the following pointwise estimate for  $\psi$ :



**Lemma 5.4.** *There exists  $C > 0$ , depending on  $g, p, K$ , such that on the regions  $\Omega_{f_0, \rho_0} \cap \mathcal{M}_\pm$ , where  $\mathcal{M}_\pm$  are as defined in (3.5), we have<sup>18</sup>*

$$(5.13) \quad \begin{aligned} \lambda^{-1} f^{n-2-p} |\mathcal{L}\psi|^2 &\geq C \lambda f^{n-2+p} |\nabla_N \psi|^2 + C f^{n-2} \rho^2 (|\nabla_V \psi|^2 + |\nabla \psi|^2) \\ &\quad + [\kappa^2 - (n-2)\kappa + \sigma - (n-1)] f^{n-2} |\psi|^2 \\ &\quad + C(\lambda f^{n-2+p} + \lambda^2 f^{n-2+2p}) |\psi|^2 + \nabla^\beta P_\beta, \end{aligned}$$

where the 1-form  $P$  satisfies, for some  $C > 0$  depending on  $g$  and  $p$ ,

$$(5.14) \quad P(\mathcal{N}) \leq C f^{n-2} \rho^2 (|\nabla_t \psi|^2 + |\nabla_\rho \psi|^2) + C(\lambda^2 + |\sigma|) f^{n-2} |\psi|^2.$$

*Proof.* Let  $Q$  be the stress-energy tensor for the wave equation with respect to  $\psi$ :

$$(5.15) \quad Q_{\alpha\beta} := \nabla_\alpha \psi^I \nabla_\beta \psi_I - \frac{1}{2} g_{\alpha\beta} \nabla^\mu \psi^I \nabla_\mu \psi_I.$$

A direct computation yields that the current

$$(5.16) \quad P_\beta^Q := Q_{\alpha\beta} S^\alpha + \frac{1}{2} h_{\xi, \zeta} \cdot \nabla_\beta |\psi|^2 - \frac{1}{2} \nabla_\beta h_{\xi, \zeta} \cdot |\psi|^2$$

satisfies the identity

$$(5.17) \quad \begin{aligned} \nabla^\beta P_\beta^Q &= \square \psi^I S_{\xi, \zeta} \psi_I - \pi_{\xi, \zeta}^{\alpha\beta} \nabla_\alpha \psi^I \nabla_\beta \psi_I \\ &\quad - S^\alpha \nabla^\beta \psi_I \mathcal{R}_{\alpha\beta} \psi^I - \frac{1}{2} \square h_{\xi, \zeta} \cdot |\psi|^2. \end{aligned}$$

The pseudoconvexity property and Theorem 3.13 imply that

$$(5.18) \quad \begin{aligned} \pi_{\xi, \zeta}^{\alpha\beta} \nabla_\alpha \psi^I \nabla_\beta \psi_I &\geq [K f^{n-2} \rho^2 + \mathcal{O}_0(f^{n-1} \rho^2)] \cdot (|\nabla_V \psi|^2 + |\nabla \psi|^2) \\ &\quad - [(n-1) f^{n-2} + \mathcal{O}_0(f^n)] \cdot |\nabla_N \psi|^2. \end{aligned}$$

Next, using (3.13), (3.14), (3.22), (5.10), and the assumption  $\zeta = \mathcal{O}(1)$ , we obtain

$$(5.19) \quad h_{\xi, \zeta} = \mathcal{O}_0(f^n), \quad \square h_{\xi, \zeta} = \mathcal{O}_0(f^n).$$

For the curvature term in (5.17), we recall (3.13) and (3.20) and expand

$$(5.20) \quad \begin{aligned} -S^\alpha \nabla^\beta \psi_I \mathcal{R}_{\alpha\beta} \psi^I &= \mathcal{O}_0(f^{n-2}) \left[ \mathcal{R}_{NV} \psi^I \nabla_V \psi_I - \sum_{X=1}^{n-1} \mathcal{R}_{NEX} \psi^I \nabla_{E_X} \psi_I \right] \\ &\geq \mathcal{O}_0(f^{n-1} \rho^2) \cdot (|\nabla_V \psi|^2 + |\nabla \psi|^2) |\psi| \\ &\geq \mathcal{O}_0(f^{n-2} \rho^4) \cdot (|\nabla_V \psi|^2 + |\nabla \psi|^2) + \mathcal{O}_0(f^n) \cdot |\psi|^2. \end{aligned}$$

Thus, applying (5.18)-(5.20) to (5.17) yields

$$(5.21) \quad \begin{aligned} \square \psi^I S_{\xi, \zeta} \psi_I &\geq \nabla^\beta P_\beta^Q + \mathcal{O}_0(f^n) \cdot |\psi|^2 - [(n-1) f^{n-2} + \mathcal{O}_0(f^n)] |\nabla_N \psi|^2 \\ &\quad + [K f^{n-2} \rho^2 + \mathcal{O}_0(f^{n-1} \rho^2)] (|\nabla_V \psi|^2 + |\nabla \psi|^2). \end{aligned}$$

Next, we expand  $\mathcal{L}$  to obtain

$$(5.22) \quad \begin{aligned} \mathcal{L}\psi &= \square \psi + 2F' f^{-n+3} \cdot \nabla_S \psi + \mathcal{A}_0 \cdot \psi, \\ \mathcal{A}_0 &= [(F')^2 + F''] \nabla^\alpha f \nabla_\alpha f + F' \square f + \sigma. \end{aligned}$$

Defining the quantities

$$(5.23) \quad P_\beta^S := \frac{1}{2} \mathcal{A} S_\beta \cdot |\psi|^2, \quad \mathcal{A} = \mathcal{A}_0 + 2F' f^{-n+3} h_{\xi, \zeta},$$

<sup>18</sup>Note in particular that all quantities are smooth in  $\Omega_{f_0, \rho_0} \cap \mathcal{M}_\pm$ .

contracting (5.22) with  $S_{\xi,\zeta}\psi$ , and applying the product rule, we see that

$$(5.24) \quad \begin{aligned} \mathcal{L}\psi^I S_{\xi,\zeta}\psi_I &= \nabla^\alpha P_\beta^S + \square\psi^I S_{\xi,\zeta}\psi_I + 2F'f^{-n+3} \cdot |\nabla_S\psi|^2 \\ &\quad + \left[ h_{\xi,\zeta}\mathcal{A}_0 - \frac{1}{2}\nabla^\alpha(S_\alpha\mathcal{A}) \right] \cdot |\psi|^2. \end{aligned}$$

By (3.13) and (5.9), we have that

$$(5.25) \quad 2F'f^{-n+3} \cdot |\nabla_S\psi|^2 = [2\kappa f^{n-2} + 2\lambda f^{n-2+p} + \lambda \cdot \mathcal{O}_0(f^n)]|\nabla_N\psi|^2.$$

Moreover, using Proposition 3.5, (5.9), and (5.19), we see that

$$(5.26) \quad \mathcal{A} = (\kappa^2 - n\kappa + \sigma) + \lambda(2\kappa - n + p)f^p + \lambda^2 f^{2p} + \lambda^2 \cdot \mathcal{O}_0(f^2).$$

Applying (3.13), (3.8), (3.14), and (5.19), we also see that

$$(5.27) \quad \begin{aligned} h_{\xi,\zeta}\mathcal{A}_0 - \frac{1}{2}\nabla^\alpha(S_\alpha\mathcal{A}) &= (\kappa^2 - n\kappa + \sigma)f^{n-2} + \frac{2-2p}{2}\lambda(2\kappa - n + p)f^{n-2+p} \\ &\quad + (1-p)\lambda^2 f^{n-2+2p} + \lambda^2 \cdot \mathcal{O}_0(f^n). \end{aligned}$$

Thus, applying (5.21) and (5.25)-(5.27) to (5.24) yields

$$(5.28) \quad \begin{aligned} \mathcal{L}\psi^I S_{\xi,\zeta}\psi_I &\geq [(2\kappa - n + 1)f^{n-2} + 2\lambda f^{n-2+p} + \lambda \cdot \mathcal{O}_0(f^n)]|\nabla_N\psi|^2 \\ &\quad + [Kf^{n-2}\rho^2 + \mathcal{O}_0(f^{n-1}\rho^2)](|\nabla_V\psi|^2 + |\nabla\psi|^2) \\ &\quad + \left[ (\kappa^2 - n\kappa + \sigma)f^{n-2} + \frac{2-p}{2}\lambda(2\kappa - n + p)f^{n-2+p} \right] |\psi|^2 \\ &\quad + [(1-p)\lambda^2 f^{n-2+2p} + \lambda^2 \cdot \mathcal{O}_0(f^n)]|\psi|^2 + \nabla^\beta(P_\beta^Q + P_\beta^S). \end{aligned}$$

Next, fix  $b, q \in \mathbb{R}$ , and observe that

$$(5.29) \quad \begin{aligned} 0 &\leq f^{q-2}|\nabla^\beta f \nabla_\beta \psi + bf \cdot \psi|^2 \\ &= f^{q-2}|\nabla^\beta f \nabla_\beta \psi|^2 + [b^2 f^q - b(q-1)f^{q-2}\nabla^\beta f \nabla_\beta f - bf^{q-1}\square f]|\psi|^2 \\ &\quad + \nabla^\beta(bf^{q-1}\nabla_\beta f \cdot |\psi|^2). \end{aligned}$$

Recalling Proposition 3.5, setting  $b = \frac{1}{2}(q-n)$ , and rearranging terms, (5.29) becomes, for any  $q$ , a pointwise weighted Hardy-type inequality:

$$(5.30) \quad \begin{aligned} f^q|\nabla_N\psi|^2 &\geq \frac{1}{4}(q-n)^2 f^q \cdot |\psi|^2 + \mathcal{O}_0(f^{q+2}) \cdot (|\nabla_N\psi|^2 + |\psi|^2) \\ &\quad - \frac{1}{2}(q-n)\nabla^\beta(f^{q-1}\nabla_\beta f \cdot |\psi|^2). \end{aligned}$$

We now apply (5.30) to the terms  $f^{n-2}|\nabla_N\psi|^2$  and  $f^{n-2+p}|\nabla_N\psi|^2$  in the right-hand side of (5.28). Defining in addition the 1-form

$$(5.31) \quad P_\beta^H := (2\kappa - n + 1)f^{n-3}\nabla_\beta f \cdot |\psi|^2 + \frac{2-p}{2}\lambda f^{n-3+p}\nabla_\beta f \cdot |\psi|^2,$$

and noting that  $2\kappa - n + 1 \geq 0$  by (5.5), we see that the above process yields

$$(5.32) \quad \begin{aligned} \mathcal{L}\psi^I S_{\xi,\zeta}\psi_I &\geq \nabla^\beta(P_\beta^Q + P_\beta^S + P_\beta^H) + [\lambda f^{n-2+p} + \lambda \cdot \mathcal{O}_0(f^n)]|\nabla_N\psi|^2 \\ &\quad + [Kf^{n-2}\rho^2 + \mathcal{O}_0(f^{n-1}\rho^2)](|\nabla_V\psi|^2 + |\nabla\psi|^2) \\ &\quad + [\kappa^2 - (n-2)\kappa + \sigma - (n-1)]f^{n-2}|\psi|^2 \\ &\quad + \frac{2-p}{2}\lambda \left( 2\kappa - n + 1 + \frac{p}{2} \right) f^{n-2+p}|\psi|^2 \\ &\quad + [(1-p)\lambda^2 f^{n-2+2p} + \lambda^2 \cdot \mathcal{O}_0(f^n)]|\psi|^2. \end{aligned}$$

Next, applying the Cauchy-Schwarz inequality, (3.13), and (5.19), we have

$$(5.33) \quad \begin{aligned} \mathcal{L}\psi^I S_{\xi,\zeta}\psi_I &\leq \lambda^{-1} f^{n-2-p} |\mathcal{L}\psi|^2 + \frac{1}{2} \lambda [f^{n-2+p} + \mathcal{O}_0(f^n)] |\nabla_N \psi|^2 \\ &\quad + \lambda \cdot \mathcal{O}_0(f^n) \cdot |\psi|^2, \end{aligned}$$

which combined with (5.32) yields

$$(5.34) \quad \begin{aligned} \lambda^{-1} f^{n-2-p} |\mathcal{L}\psi|^2 &\geq \nabla^\beta (P_\beta^Q + P_\beta^S + P_\beta^H) \\ &\quad + \left[ \frac{1}{2} \lambda f^{n-2+p} + \lambda \cdot \mathcal{O}_0(f^n) \right] |\nabla_N \psi|^2 \\ &\quad + [K f^{n-2} \rho^2 + \mathcal{O}_0(f^{n-1} \rho^2)] (|\nabla_V \psi|^2 + |\nabla \psi|^2) \\ &\quad + [\kappa^2 - (n-2)\kappa + \sigma - (n-1)] f^{n-2} |\psi|^2 \\ &\quad + \frac{2-p}{2} \lambda \left( 2\kappa - n + 1 + \frac{p}{2} \right) f^{n-2+p} |\psi|^2 \\ &\quad + [(1-p)\lambda^2 f^{n-2+2p} + \lambda^2 \cdot \mathcal{O}_0(f^n)] |\psi|^2. \end{aligned}$$

Recalling that  $f \ll_{g,l,p,K} 1$  in (5.34) and setting

$$(5.35) \quad P := P^Q + P^S + P^H$$

results in the first identity (5.13). To complete the proof of Lemma 5.4, it remains to show that  $P$ , as defined in (5.35), satisfies (5.14).

Applying (5.12) to (5.23) and (5.31), we see that

$$(5.36) \quad \begin{aligned} P^S(\mathcal{N}) + P^H(\mathcal{N}) &= \frac{1}{2} \mathcal{A} f^{n-3} \rho \partial_\rho f \cdot |\psi|^2 + (2\kappa - n + 1) f^{n-2} \cdot |\psi|^2 \\ &\quad + \frac{2-p}{2} \lambda f^{n-2+p} \cdot |\psi|^2 \\ &\leq \mathcal{C}(\lambda^2 + |\sigma|) f^{n-2} \cdot |\psi|^2, \end{aligned}$$

for some  $\mathcal{C} > 0$ . Next, for  $P^Q$ , we expand using (5.16):

$$(5.37) \quad \begin{aligned} P^Q(\mathcal{N}) &= \rho \cdot \nabla_S \psi^I \nabla_\rho \psi_I - \frac{1}{2} \rho \cdot g(S, \partial_\rho) \cdot \nabla^\mu \psi^I \nabla_\mu \psi_I \\ &\quad + h_{\xi,\zeta} \cdot \psi^I \nabla_N \psi_I - \frac{1}{2} \rho \partial_\rho h_{\xi,\zeta} \cdot |\psi|^2. \end{aligned}$$

Using Proposition 2.10, (3.8), (3.14), (5.12), and (5.19), we see that, for some  $\mathcal{C} > 0$ ,

$$(5.38) \quad P^Q(\mathcal{N}) \leq \mathcal{C} f^{n-2} \rho^2 (|\nabla_\rho \psi|^2 + |\nabla_t \psi|^2) + \mathcal{C} f^n \cdot |\psi|^2.$$

In both (5.36) and (5.38), we used that  $f \ll_{g,l,p,K} 1$ . Furthermore, similar to [7], we used that the leading-order  $|\nabla \psi|^2$ -term in the expansion of  $P^Q(\mathcal{N})$  is negative and hence can be omitted. Finally, summing (5.36) and (5.38) results in the bound (5.14) and completes the proof of the lemma.  $\square$

We now convert Lemma 5.4 into estimates for  $\phi$ :

**Lemma 5.5.** *There exists  $\mathcal{C} > 0$ , depending on  $g, p, K$ , such that*

$$(5.39) \quad \begin{aligned} \lambda^{-1} f^{-p} E_{\kappa,\lambda}^p |(\square + \sigma)\phi|^2 &\geq C E_{\kappa,\lambda}^p (\rho^4 |\nabla_t \phi|^2 + \rho^4 |\nabla_\rho \phi|^2 + \rho^2 |\nabla \phi|^2) \\ &\quad + E_{\kappa,\lambda}^p [\kappa^2 - (n-2)\kappa + \sigma - (n-1)] |\phi|^2 \\ &\quad + C \lambda^2 E_{\kappa,\lambda}^p f^{2p} |\phi|^2 + \nabla^\beta P_\beta, \end{aligned}$$

on the regions  $\Omega_{f_0, \rho_0} \cap \mathcal{M}_\pm$ , where

$$(5.40) \quad E_{\kappa, \lambda}^p := e^{-2F} f^{n-2} = f^{n-2-2\kappa} e^{\frac{-2\lambda f^p}{p}},$$

and where  $P$ , from (5.14), satisfies, for some  $C > 0$  depending on  $g$  and  $p$ ,

$$(5.41) \quad \rho^{-n} \cdot P(\mathcal{N}) \leq C[|\nabla_t(\rho^{-\kappa}\phi)|^2 + |\nabla_\rho(\rho^{-\kappa}\phi)|^2] + C(\lambda^2 + |\sigma|)|\rho^{-\kappa-1}\phi|^2.$$

*Proof.* From (5.13), we use the largeness of  $\lambda$  and the smallness of  $f$  to obtain

$$(5.42) \quad \begin{aligned} \lambda^{-1} f^{n-2-p} |\mathcal{L}\psi|^2 &\geq C f^{n-2+2p} |\nabla_N \psi|^2 + C f^{n-2} \rho^2 (|\nabla_V \psi|^2 + |\nabla \psi|^2) \\ &\quad + [\kappa^2 - (n-2)\kappa + \sigma - (n-1)] f^{n-2} |\psi|^2 \\ &\quad + C \lambda^2 f^{n-2+2p} |\psi|^2 + \nabla^\beta P_\beta. \end{aligned}$$

By (3.8), (5.9), and the assumption  $\lambda \geq 1 + \kappa$ ,

$$(5.43) \quad e^{-2F} |\nabla_N \phi|^2 = |\nabla_N \psi + F' \nabla_N f \cdot \psi|^2 \lesssim |\nabla_N \psi|^2 + \lambda^2 |\psi|^2.$$

As a result, applying (5.40) and (5.43) to (5.42) yields

$$(5.44) \quad \begin{aligned} \lambda^{-1} f^{-p} E_{\kappa, \lambda}^p |(\square + \sigma)\phi|^2 &\geq C E_{\kappa, \lambda}^p (\rho^2 |\nabla_V \phi|^2 + \rho^2 |\nabla \phi|^2 + f^{2p} |\nabla_N \phi|^2) \\ &\quad + E_{\kappa, \lambda}^p [\kappa^2 - (n-2)\kappa + \sigma - (n-1)] |\phi|^2 \\ &\quad + C \lambda^2 E_{\kappa, \lambda}^p f^{2p} |\phi|^2 + \nabla^\beta P_\beta, \end{aligned}$$

for some (possibly different)  $C > 0$ . Moreover, by (3.19),

$$(5.45) \quad \begin{aligned} \rho^4 |\nabla_\rho \phi|^2 &\lesssim \rho^2 |\nabla_N \phi|^2 + f^2 \rho^2 |\nabla_V \phi|^2 + \rho^6 |\nabla \phi|^2, \\ \rho^4 |\nabla_t \phi|^2 &\lesssim \rho^2 |\nabla_V \phi|^2 + f^2 \rho^2 |\nabla_N \phi|^2 + \rho^6 |\nabla \phi|^2, \end{aligned}$$

which when combined with (5.44) results in (5.39).

For (5.41), we first define the shorthand

$$(5.46) \quad \mathcal{E}_{p, \lambda} := e^{-\frac{\lambda f^p}{p}} \leq 1,$$

and we note that

$$(5.47) \quad |\partial_\rho \mathcal{E}_{p, \lambda}| \lesssim \lambda f^p \rho^{-1} \mathcal{E}_{p, \lambda}, \quad |\partial_t \mathcal{E}_{p, \lambda}| \lesssim \lambda f^{p+1} \rho^{-1} \mathcal{E}_{p, \lambda}.$$

Noting that  $2\kappa \geq n-1$  by (5.5), we see from (5.46) and (5.47) that

$$(5.48) \quad \begin{aligned} f^{n-2} \rho^{-n} \cdot |\psi|^2 &\leq |\rho^{-\kappa-1} \phi|^2, \\ f^{n-2} \rho^{-n+2} |\nabla_\rho \psi|^2 &\lesssim |\nabla_\rho(\rho^{-\kappa}\phi)|^2 + \lambda^2 |\rho^{-\kappa-1} \phi|^2, \\ f^{n-2} \rho^{-n+2} |\nabla_t \psi|^2 &\lesssim |\nabla_t(\rho^{-\kappa}\phi)|^2 + \lambda^2 |\rho^{-\kappa-1} \phi|^2. \end{aligned}$$

Combining (5.14) with (5.48) yields (5.41).  $\square$

**5.1.2. Integral Estimates.** It remains to integrate (5.39) over  $\Omega_{f_0, \rho_0}$  and apply the divergence theorem. Compared to [7], the process here is a bit more complex, since we must account for the lack of smoothness at  $t = \frac{\pi T}{2}$ :

- First, we integrate (5.39) over  $\Omega_{f_0, \rho_0} \cap \mathcal{M}_+$  and apply the divergence theorem. The term  $\nabla^\alpha P_\alpha$  yields boundary terms on  $\{\rho = \rho_0\}$  and  $\{t = \frac{\pi T}{2}\}$ .
- We also integrate (5.39) over  $\Omega_{f_0, \rho_0} \cap \mathcal{M}_-$ , which yields corresponding boundary terms on  $\{\rho = \rho_0\}$  and  $\{t = \frac{\pi T}{2}\}$ .

(On the other hand, we do not obtain boundary terms on  $\{f = f_0\}$ , since we assumed both  $\phi$  and  $\nabla\phi$  vanished on  $\{f = f_0\}$ .)

Summing the two inequalities obtain above, we obtain that

$$\begin{aligned}
 (5.49) \quad & \lambda^{-1} \int_{\Omega_{f_0, \rho_0}} f^{-p} E_{\kappa, \lambda}^p |(\square + \sigma)\phi|^2 + \int_{\{\rho = \rho_0\}} P(\mathcal{N}) \\
 & \geq C \int_{\Omega_{f_0, \rho_0}} E_{\kappa, \lambda}^p (\rho^4 |\nabla_t \phi|^2 + \rho^4 |\nabla_\rho \phi|^2 + \rho^2 |\nabla \phi|^2) \\
 & \quad + \int_{\Omega_{f_0, \rho_0}} E_{\kappa, \lambda}^p [\kappa^2 - (n-2)\kappa + \sigma - (n-1)] |\phi|^2 \\
 & \quad + C\lambda^2 \int_{\Omega_{f_0, \rho_0}} E_{\kappa, \lambda}^p f^{2p} |\phi|^2 + \lim_{\tau \rightarrow \frac{\pi T}{2} +} \int_{\Omega_{f_0, \rho_0} \cap \{t = \tau\}} P(\mathcal{T}) \\
 & \quad - \lim_{\tau \rightarrow \frac{\pi T}{2} -} \int_{\Omega_{f_0, \rho_0} \cap \{t = \tau\}} P(\mathcal{T}),
 \end{aligned}$$

where  $\mathcal{T}$  denotes the future-pointing ( $g$ -)unit normal on the level sets of  $t$ . Rewriting the integral over  $\{\rho = \rho_0\}$  in (5.49) (with respect to the induced metric) in terms of the volume form from  $\mathfrak{g}$  and then applying (5.41), we have

$$\begin{aligned}
 (5.50) \quad & \int_{\{\rho = \rho_0\}} P(\mathcal{N}) \leq C' \int_{\{\rho = \rho_0\}} \rho^{-n} P(\mathcal{N}) \cdot d\mathfrak{g} \\
 & \leq C \int_{\{\rho = \rho_0\}} [|\nabla_t(\rho^{-\kappa}\phi)|^2 + |\nabla_\rho(\rho^{-\kappa}\phi)|^2] d\mathfrak{g} \\
 & \quad + C(\lambda^2 + |\sigma|) \int_{\{\rho = \rho_0\}} |\rho^{-\kappa-1}\phi|^2 d\mathfrak{g},
 \end{aligned}$$

for appropriate constants  $C'$  and  $C$ .

It remains to control the spacelike boundary terms

$$(5.51) \quad \mathcal{Y} := \lim_{\tau \rightarrow \frac{\pi T}{2} +} \int_{\Omega_{f_0, \rho_0} \cap \{t = \tau\}} P(\mathcal{T}) - \lim_{\tau \rightarrow \frac{\pi T}{2} -} \int_{\Omega_{f_0, \rho_0} \cap \{t = \tau\}} P(\mathcal{T}),$$

which we show is an error term that can be absorbed by the remaining terms. To see this, we examine the various terms within  $P(\mathcal{T})$ ; see (5.16), (5.23), (5.31), and (5.35). Since  $f$  and  $\eta$  are  $C^2$  (in particular,  $w_{\xi, \zeta}$  and  $h_{\xi, \zeta}$  are both continuous at  $\{t = \frac{\pi T}{2}\}$ ), it follows that all the terms in  $P(\mathcal{T})$  in the limits  $t \rightarrow \frac{\pi T}{2} +$  and  $t \rightarrow \frac{\pi T}{2} -$  coincide, except for the term  $-\frac{1}{2}\mathcal{T}h_{\xi, \zeta} \cdot |\psi|^2$ , i.e., the last term in (5.16).

Since both  $h_{\xi, \zeta}|_{\Omega_{f_0, \rho_0} \cap \mathcal{M}_\pm}$  extend smoothly to  $\{t = \frac{\pi T}{2}\}$ , we can then bound

$$\begin{aligned}
 (5.52) \quad & \mathcal{Y} \lesssim \lim_{\tau \rightarrow \frac{\pi T}{2} +} \int_{\Omega_{f_0, \rho_0} \cap \{t = \tau\}} |\mathcal{T}h_{\xi, \zeta}| |\psi|^2 + \lim_{\tau \rightarrow \frac{\pi T}{2} -} \int_{\Omega_{f_0, \rho_0} \cap \{t = \tau\}} |\mathcal{T}h_{\xi, \zeta}| |\psi|^2 \\
 & \lesssim \int_{\Omega_{f_0, \rho_0} \cap \{t = \frac{\pi T}{2}\}} f^n |\psi|^2,
 \end{aligned}$$

where in the last step, we applied (3.14) and (5.19). Using that  $\eta \simeq 1$ , and hence  $f \simeq \rho$ , near  $\{t = \frac{\pi T}{2}\}$ , and applying Proposition 2.10 to expand the volume form on  $\{t = \frac{\pi T}{2}\}$  in the usual coordinates, we obtain that

$$(5.53) \quad \mathcal{Y} \lesssim \sum_{\varphi} \int_{\Omega_{f_0, \rho_0} \cap \{t = \frac{\pi T}{2}\}} f^{n-2} \cdot \rho^2 |\psi|^2 \cdot \rho^{-n} d\rho dx^1 \cdot dx^{n-1}$$

$$\lesssim \sum_{\varphi} \int_{\Omega_{f_0, \rho_0} \cap \{t = \frac{\pi T}{2}\}} \rho^2 E_{p, \lambda}^{\kappa} |\phi|^2 \cdot \rho^{-n} d\rho dx^1 \cdot dx^{n-1},$$

where the summation is over coordinate systems  $\varphi = (x^1, \dots, x^{n-1})$  on  $\mathcal{S}$  comprising the definition of admissible AdS segments, and where  $\mathcal{E}_{p, \lambda}$  is as in (5.46).

Next, we take sufficiently small  $A > 0$ , apply the fundamental theorem of calculus, and convert the coordinate exterior form to a spacetime volume form:

$$(5.54) \quad \mathcal{Y} \lesssim_A \sum_{\varphi} \int_{\Omega_{f_0, \rho_0} \cap \{|t - \frac{\pi T}{2}| < A\}} \rho^2 \partial_t (E_{p, \lambda}^{\kappa} |\phi|^2) \cdot \rho^{-n} d\rho dt dx^1 \cdot dx^{n-1} \\ \lesssim \int_{\Omega_{f_0, \rho_0} \cap \{|t - \frac{\pi T}{2}| < A\}} \rho^3 \partial_t (E_{p, \lambda}^{\kappa} |\phi|^2).$$

Recalling again that  $f \simeq \rho$  here in our region of integration, we conclude

$$(5.55) \quad \mathcal{Y} \lesssim_A \int_{\Omega_{f_0, \rho_0} \cap \{|t - \frac{\pi T}{2}| < A\}} E_{\kappa, \lambda}^p \cdot \rho^3 (|\phi| |\nabla_t \phi| + \lambda |\phi|^2) \\ \lesssim \int_{\Omega_{f_0, \rho_0} \cap \{|t - \frac{\pi T}{2}| < A\}} E_{\kappa, \lambda}^p \cdot (\rho^5 |\nabla_t \phi|^2 + \lambda \rho |\phi|^2).$$

Note the right-hand side of (5.55), and hence  $\mathcal{Y}$ , can be absorbed into the first and third terms on the right-hand side of (5.49), as long as  $\rho \lesssim f_0$  is small. Finally, combining this with (5.49) and (5.50) results in (5.7) and proves Theorem 5.3.

**5.2. Proof of Theorem 5.2.** This is analogous to the corresponding proof in [7], hence we give only an abridged summary. Assume the hypotheses of Theorem 5.2, and let  $\bar{\chi} : [0, f_0] \rightarrow [0, 1]$  denote a smooth cut-off function satisfying

$$(5.56) \quad \bar{\chi}(s) = \begin{cases} 1 & 0 \leq s \leq \frac{f_0}{2}, \\ 0 & s > \frac{3f_0}{4}. \end{cases}$$

Letting  $\chi := \bar{\chi} \circ f$  and letting  $\prime$  denote differentiation with respect to  $f$ , we have

$$(5.57) \quad (\square_g + \sigma)(\chi \cdot \phi) = \chi'(2\nabla^\alpha f \nabla_\alpha \phi + \square_g f \cdot \phi) + \chi'' \nabla^\alpha f \nabla_\alpha f \cdot \phi \\ + \chi(\square_g \phi + \sigma \phi).$$

Note that  $\chi'$  and  $\chi''$  are supported in  $[\frac{1}{2}f_0, \frac{3}{4}f_0]$ . Letting  $\mathcal{F}$  denote the right-hand side of (5.57), then applying (3.8), (3.9), and (5.1), we compute<sup>19</sup>

$$(5.58) \quad |\mathcal{F}|^2 \begin{cases} \lesssim_{g, f_0} (\rho^2 |\nabla_\rho \phi|^2 + \rho^2 |\nabla_t \phi|^2 + \rho^{2+p} |\nabla \phi|^2 + |\phi|^2) & \frac{f_0}{2} \leq f \leq \frac{3f_0}{4}, \\ \lesssim \rho^p (\rho^4 |\nabla_t \phi|^2 + \rho^4 |\nabla_\rho \phi|^2 + \rho^2 |\nabla \phi|^2 + \rho^{2p} |\phi|^2) & 0 \leq f \leq \frac{f_0}{2}. \end{cases}$$

Recall the region  $\Omega_{f_0, \rho_0}$  from (5.6), for  $\rho_0 \ll f_0$ , and let

$$(5.59) \quad \Omega_i = \Omega_{f_0, \rho_0} \cap \left\{ f < \frac{f_0}{2} \right\}, \quad \Omega_e = \Omega_{f_0, \rho_0} \cap \left\{ \frac{f_0}{2} < f < \frac{3f_0}{4} \right\}.$$

We now apply (5.7) to  $\bar{\phi} := \chi \phi$ , with  $\kappa$  given by (5.2).<sup>20</sup> Recalling (5.58), then the left-hand side  $L$  of (5.7) can then be estimated

$$(5.60) \quad L \lesssim \int_{\Omega_e} E_{\kappa, \lambda}^p f^{-p} (\rho^2 |\nabla_\rho \phi|^2 + \rho^2 |\nabla_t \phi|^2 + \rho^{2+p} |\nabla \phi|^2 + |\phi|^2)$$

<sup>19</sup>In the case  $\frac{1}{2}f_0 \leq f \leq \frac{3}{4}f_0$ , we also used that  $f \simeq_{f_0} 1$ .

<sup>20</sup>In particular,  $\kappa$  satisfies (5.5).

$$\begin{aligned}
 & + \int_{\Omega_i} E_{\kappa,\lambda}^p f^{-p} \rho^p (\rho^4 |\nabla_t \phi|^2 + \rho^4 |\nabla_\rho \phi|^2 + \rho^2 |\nabla \phi|^2 + \rho^{2p} |\phi|^2) \\
 & + \lambda (\lambda^2 + |\sigma|) \int_{\{\rho=\rho_0\}} [|\nabla_t(\rho^{-\kappa} \bar{\phi})|^2 + |\nabla_\rho(\rho^{-\kappa} \bar{\phi})|^2 + |\rho^{-\kappa-1} \bar{\phi}|^2] d\mathbf{g} \\
 & := L_1 + L_2 + L_3,
 \end{aligned}$$

where  $E_{\kappa,\lambda}^p$  is as defined in (5.40), while the right-hand side  $R$  of (5.7) satisfies

$$\begin{aligned}
 (5.61) \quad R & \gtrsim_{g,p,K} \lambda^3 \int_{\Omega_i} E_{\kappa,\lambda}^p \rho^{2p} |\phi|^2 + \lambda \int_{\Omega_i} E_{\kappa,\lambda}^p (\rho^4 |\nabla_t \phi|^2 + \rho^4 |\nabla_\rho \phi|^2 + \rho^2 |\nabla \psi|^2) \\
 & := R_1 + R_2.
 \end{aligned}$$

In particular, note that all terms on the right-hand side of (5.7) are non-negative.

Since  $\rho \lesssim f$ , we can absorb  $L_2$  into  $R_1 + R_2$  when  $\lambda$  is large enough, so that

$$(5.62) \quad L_1 + L_3 \gtrsim_{g,p,K} R_1 + R_2.$$

Next, by (5.2), along with the bounds

$$|\partial_\rho \chi| + |\partial_t \chi| \lesssim_{g,f_0} \rho^{-1},$$

we see that  $L_3 \rightarrow 0$  when  $\rho_0 \searrow 0$ . We can also eliminate the  $E_{\kappa,\lambda}^p$ 's in (5.62), since

$$E_{\kappa,\lambda}^p \begin{cases} \leq \left(\frac{f_0}{2}\right)^{n-2-2\kappa} e^{\frac{-2\lambda(\frac{f_0}{2})^p}{p}} & \frac{f_0}{2} \leq f \leq \frac{3f_0}{4}, \\ \geq \left(\frac{f_0}{2}\right)^{n-2-2\kappa} e^{\frac{-2\lambda(\frac{f_0}{2})^p}{p}} & f < \frac{f_0}{2}. \end{cases}$$

As a result, we obtain that for large  $\lambda$ ,

$$\begin{aligned}
 (5.63) \quad & \int_{\{\frac{f_0}{2} < f < \frac{3f_0}{4}\}} (|\phi|^2 + \rho^2 |\nabla_\rho \phi|^2 + \rho^2 |\nabla_t \phi|^2 + \rho^{2+p} |\nabla \phi|^2) \\
 & \gtrsim \lambda \int_{\{f < \frac{f_0}{2}\}} \rho^{2p} |\phi|^2.
 \end{aligned}$$

Finally, one can see that (5.2) and (5.3) imply that the left-hand side of (5.63) is finite. As a result, taking  $\lambda \nearrow \infty$  yields that  $\phi \equiv 0$  on  $\{f < \frac{f_0}{2}\}$ .

**5.3. The Borderline Case.** We conclude this section by stating and proving the analogue of Theorem 5.2 in the static borderline case detailed in Section 4.

**Theorem 5.6.** *Consider an  $(n+1)$ -dimensional borderline FG-aAdS segment*

$$(\mathcal{M}, g), \quad \mathcal{M} = (0, \rho_0) \times (0, \pi) \times \mathcal{S},$$

*and consider on  $(\mathcal{M}, g)$  the wave equation (5.1), for some  $l \geq 0$  and  $\sigma \in \mathbb{R}$ . Furthermore, assume that the following properties hold:*

- (1) *The pseudoconvexity property (see Definition 3.12) holds on  $\mathcal{I}$ .*
- (2) *There exist  $0 < p \ll 1$  and  $C > 0$  such that for any  $\phi \in \Gamma_{\mathcal{I}}^0 \mathcal{M}$ ,*

$$(5.64) \quad |\mathcal{G}(\phi, \nabla \phi)|^2 \leq C \rho^p [\rho^5 |\nabla_\rho \phi|^2 + \rho^5 |\nabla_t \phi|^2 + \rho^3 |\nabla \phi|^2 + \rho^{2p} |\phi|^2].$$

*Then, the local unique continuation property holds on  $(\mathcal{M}, g)$  for (5.1).*

**Remark.** By inspecting the proof of Theorem 5.6, one can observe that in the borderline case, the finiteness condition (5.2) can be replaced by

$$\int_{\mathcal{M}} \rho^{3+p} |\nabla \phi|^2 < \infty, \quad 0 < p \ll 1.$$

The remainder of this subsection is dedicated to proving Theorem 5.6. As a vast majority of the proof is completely analogous to that of Theorem 5.2, we will focus only on the differences between the proofs and leave other details to the reader.

The main step of the proof is the derivation of Carleman estimate in the borderline setting, which we accomplish below. Since the process of going from the borderline Carleman estimate to the conclusions of Theorem 5.6 is completely analogous to the argument in Section 5.2, in the remaining development, we will restrict our discussions to only the Carleman estimate.

**5.3.1. The Borderline Carleman Estimate.** We now state and prove our corresponding Carleman estimate in the borderline setting:

**Theorem 5.7.** Consider an  $(n+1)$ -dimensional borderline FG-aAdS segment

$$(\mathcal{M}, g), \quad \mathcal{M} = (0, \rho_0) \times (0, T\pi) \times \mathcal{S},$$

and assume the borderline pseudoconvexity property holds on  $\mathcal{I}$ , with parameters  $K, \zeta$ . Fix also  $l \geq 0$ , constants  $p, \kappa \in \mathbb{R}$  satisfying (5.5), and  $0 < \rho_0 \ll f_0 \ll_{g,l,p,K} 1$ .

Then, there exist constants  $C, \mathcal{C} > 0$ , depending on  $g, p$ , and  $K$ , such that for any  $\sigma \in \mathbb{R}$  and  $\lambda \in [1 + \kappa, \infty)$ , and for any  $\phi \in \Gamma_{\mathcal{I}}^0 \mathcal{M}$  such that both  $\phi$  and  $\nabla \phi$  vanish on  $\{\hat{f} = f_0\}$ , the following inequality holds:

$$\begin{aligned} (5.65) \quad & \int_{\Omega_{f_0, \rho_0}} \hat{f}^{n-2-2\kappa} e^{\frac{-2\lambda \hat{f} p}{p}} \hat{f}^{-p} |(\square + \sigma)\phi|^2 \\ & + C\lambda(\lambda^2 + |\sigma|) \int_{\{\rho=\rho_0\}} [|\nabla_t(\rho^{-\kappa}\phi)|^2 + |\nabla_\rho(\rho^{-\kappa}\phi)|^2 + |\rho^{-\kappa-1}\phi|^2] d\mathbf{g} \\ & \geq C\lambda \int_{\Omega_{f_0, \rho_0}} \hat{f}^{n-2-2\kappa} e^{\frac{-2\lambda \hat{f} p}{p}} (\rho^5 |\nabla_t \phi|^2 + \rho^5 |\nabla_\rho \phi|^2 + \rho^3 |\nabla \phi|^2) \\ & + \lambda[\kappa^2 - (n-2)\kappa + \sigma - (n-1)] \int_{\Omega_{f_0, \rho_0}} \hat{f}^{n-2-2\kappa} e^{\frac{-2\lambda \hat{f} p}{p}} |\phi|^2 \\ & + C\lambda^3 \int_{\Omega_{f_0, \rho_0}} \hat{f}^{n-2-2\kappa} e^{\frac{-2\lambda \hat{f} p}{p}} \hat{f}^{2p} |\phi|^2, \end{aligned}$$

where  $\Omega_{f_0, \rho_0} := \{\hat{f} < f_0, \rho > \rho_0\}$ .

*Proof.* We adopt analogues of the notations established in Section 5.1:

- We now conjugate using  $\hat{f}$  instead of  $f$ :

$$(5.66) \quad F := \kappa \cdot \log \hat{f} + \lambda p^{-1} \hat{f}^p, \quad \psi := e^{-F} \phi.$$

- Moreover, we define the borderline analogues of (5.10):

$$(5.67) \quad \hat{S}_\zeta \psi := \nabla_{\hat{S}} \psi + \hat{h}_\zeta \psi, \quad \hat{h}_\zeta := \hat{f}^{n-3} \hat{w}_\zeta + \frac{1}{2} \nabla^\alpha \hat{S}_\alpha.$$

- Finally, we define  $\mathcal{L}$  and  $\mathcal{N}$  as in (5.11) and (5.12), respectively.



The key step is the following analogue of Lemma 5.4:

$$\begin{aligned}
 (5.68) \quad \lambda^{-1} \hat{f}^{n-2-p} |\mathcal{L}\psi|^2 &\geq C \lambda \hat{f}^{n-2+p} |\nabla_{\hat{N}} \psi|^2 + C \hat{f}^{n-2} \rho^3 (|\nabla_{\hat{V}} \psi|^2 + |\nabla \psi|^2) \\
 &\quad + [\kappa^2 - (n-2)\kappa + \sigma - (n-1)] \hat{f}^{n-2} |\psi|^2 \\
 &\quad + C(\lambda \hat{f}^{n-2+p} + \lambda^2 \hat{f}^{n-2+2p}) |\psi|^2 + \nabla^\beta P_\beta, \\
 P(\mathcal{N}) &\leq C \hat{f}^{n-2} \rho^2 (|\nabla_t \psi|^2 + |\nabla_\rho \psi|^2) + \mathcal{C}(\lambda^2 + |\sigma|) \hat{f}^{n-2} |\psi|^2.
 \end{aligned}$$

Note that the estimate now holds on all of  $\mathcal{M} \cap \Omega_{f_0, \rho_0}$ , since  $\hat{f}$  is now everywhere smooth. As before,  $C > 0$  depends on  $g, p, K$ , while  $\mathcal{C} > 0$  depends on  $g$  and  $p$ .

The proof is almost identical to that of Lemma 5.4. Besides replacing  $f$  by  $\hat{f}$ , the only other differences are in the powers of  $\rho$  in some asymptotic expansions:

- Here, the pseudoconvexity of the level sets of  $\hat{f}$  is more degenerate than that of  $f$ . By Theorem 3.13, we deduce that the analogue of (5.18) is

$$\begin{aligned}
 (5.69) \quad \hat{\pi}_\zeta^{\alpha\beta} \nabla_\alpha \psi^I \nabla_\beta \psi_I &\geq [K \hat{f}^{n-2} \rho^3 + \mathcal{O}_0(\hat{f}^{n-1} \rho^3)] \cdot (|\nabla_{\hat{V}} \psi|^2 + |\nabla \psi|^2) \\
 &\quad - [(n-1) \hat{f}^{n-2} + \mathcal{O}_0(\hat{f}^n)] \cdot |\nabla_{\hat{N}} \psi|^2.
 \end{aligned}$$

- A similar difference lies in the curvature expansion (in the tensorial case). More specifically, by (4.18), the analogue of (5.20) is

$$\begin{aligned}
 (5.70) \quad -\hat{S}^\alpha \nabla^\beta \psi_I \mathcal{R}_{\alpha\beta} \psi^I &= \mathcal{O}_0(\hat{f}^{n-2}) \left[ \mathcal{R}_{\hat{N}\hat{V}} \psi^I \nabla_{\hat{V}} \psi_I - \sum_{X=1}^{n-1} \mathcal{R}_{\hat{N}\hat{E}_X} \psi^I \nabla_{\hat{E}_X} \psi_I \right] \\
 &\geq \mathcal{O}_0(\hat{f}^{n-2} \rho^4) \cdot (|\nabla_{\hat{V}} \psi| + |\nabla \psi|) |\psi| \\
 &\geq \mathcal{O}_0(\hat{f}^{n-2} \rho^4) \cdot (|\nabla_{\hat{V}} \psi|^2 + |\nabla \psi|^2) + \mathcal{O}_0(\hat{f}^n) \cdot |\psi|^2.
 \end{aligned}$$

The remaining steps, being essentially the same as before, are left to the reader.

From (5.68), we express  $\psi$  in terms of  $\phi$ , and we expand the frame elements  $\hat{N}$ ,  $\hat{V}$ ,  $\hat{E}_X$  in terms of coordinate derivatives. The derivation is identical to the proof of Lemma 5.5; the only real difference is that due to the extra power of  $\rho$  in (5.69) and (5.70), we inherit an extra power of  $\rho$  in the coordinate derivatives:

$$\begin{aligned}
 (5.71) \quad \lambda^{-1} \hat{f}^{-p} E_{\kappa, \lambda}^p |(\square + \sigma)\phi|^2 &\geq C \hat{E}_{\kappa, \lambda}^p (\rho^5 |\nabla_t \phi|^2 + \rho^5 |\nabla_\rho \phi|^2 + \rho^3 |\nabla \phi|^2) \\
 &\quad + \hat{E}_{\kappa, \lambda}^p [\kappa^2 - (n-2)\kappa + \sigma - (n-1)] |\phi|^2 \\
 &\quad + C \lambda^2 \hat{E}_{\kappa, \lambda}^p \hat{f}^{2p} |\phi|^2 + \nabla^\beta P_\beta, \\
 \rho^{-n} \cdot P(\mathcal{N}) &\leq C [|\nabla_t(\rho^{-\kappa} \phi)|^2 + |\nabla_\rho(\rho^{-\kappa} \phi)|^2] \\
 &\quad + \mathcal{C}(\lambda^2 + |\sigma|) |\rho^{-\kappa-1} \phi|^2.
 \end{aligned}$$

Here,  $C$  and  $\mathcal{C}$  satisfy the same assumptions as above, and

$$(5.72) \quad \hat{E}_{\kappa, \lambda}^p := \hat{f}^{n-2-2\kappa} e^{\frac{-2\lambda \hat{f}^p}{p}}.$$

The final step is to integrate (5.71) and apply the divergence theorem, which results in (5.65) and completes the proof. This process is similar to that in Section 5.1.2 but is simpler since all quantities here are smooth on  $\mathcal{M} \cap \Omega_{f_0, \rho_0}$ . Thus, here we do not have to deal with any discontinuities on  $\{t = \frac{\pi}{2}\}$ .  $\square$

## APPENDIX A. REDUCTION TO THE FEFFERMAN-GRAHAM GAUGE

In this appendix, we prove the following theorem, which demonstrates that any admissible aAdS segment can be rewritten as an admissible FG-aAdS segment via an appropriate change of coordinates.

**Theorem A.1.** *Let  $(\underline{\mathcal{M}}, g)$  be an admissible aAdS segment, with*

$$\underline{\mathcal{M}} := (0, \underline{\rho}_0) \times \underline{\mathcal{I}} := (0, \underline{\rho}_0) \times (\underline{T}_-, \underline{T}_+) \times \mathcal{S},$$

*and with induced AdS-type boundary  $(\underline{\mathcal{I}}, \mathring{\mathfrak{g}})$ . Then, the following statements hold:*

- (1) *Given any  $\underline{T}_- < T_- < T_+ < \underline{T}_+$ , there exists  $\rho_0 > 0$  and an isometry  $\Phi$  between an open subset  $\underline{\mathcal{D}}$  of  $\underline{\mathcal{M}}$  and an admissible FG-aAdS segment*

$$(\mathcal{M}, g), \quad \mathcal{M} := (0, \rho_0) \times \mathcal{I} := (0, \rho_0) \times (T_-, T_+) \times \mathcal{S},$$

*with induced AdS-type boundary  $(\mathcal{I}, \mathring{\mathfrak{g}}|_{\mathcal{I}})$ .*

- (2) *Letting  $\underline{\rho}$  and  $\rho$  denote the projections to the first components of  $\underline{\mathcal{M}}$  and  $\mathcal{M}$ , respectively, then the following comparison holds:<sup>21</sup>*

$$(A.1) \quad \rho \simeq \underline{\rho} \circ \Phi.$$

*In particular,  $\Phi^{-1}$  maps the conformal boundary  $\mathcal{I}$  of  $\mathcal{M}$  to  $\underline{\mathcal{I}}$ .*

**Remark.** *In practice, the aAdS segments on which our unique continuation theorems apply are subsets of larger spacetimes. Thus, the (arbitrarily small) shortening of the time interval in Theorem A.1 would not result in any loss of generality.*

The remainder of this appendix is dedicated to the proof of Theorem A.1. Let  $\underline{\rho}$  and  $\underline{t}$  be the usual projections on  $\underline{\mathcal{M}}$ , and let  $\underline{h} := \underline{\rho}^2 g$ . Recall from (2.7) that

$$(A.2) \quad \underline{h} = (1 + \underline{\rho}^2 \bar{g}_{\rho\rho}) d\underline{\rho}^2 + (-d\underline{t}^2 + \mathring{g}_{AB} d\underline{x}^A d\underline{x}^B + \underline{\rho}^2 \bar{g}_{ab} d\underline{x}^a d\underline{x}^b) + \mathcal{O}(\underline{\rho}^3) \cdot d\underline{x}^\alpha d\underline{x}^\beta.$$

A few notational clarifications are in order here:

- To avoid clutter, we do not underline symbols in superscript and subscript indices. For  $(\underline{\mathcal{M}}, g)$ -related quantities (e.g.,  $\underline{h}$ ,  $\mathring{g}$ ,  $\bar{g}$ ), indices are understood to be with respect to  $(\underline{\rho}, \underline{t}, \underline{x}^A)$ -coordinates.
- We define the class  $\mathcal{O}(\zeta)$  to be as in Definition 2.4, except now to be with respect to the underlined  $(\underline{\rho}, \underline{t}, \underline{x}^A)$ -coordinate systems.

Let  $\underline{\Lambda}_{\alpha\beta}^\mu$  denote the Christoffel symbols for  $\underline{h}$ , in the  $(\underline{\rho}, \underline{t}, \underline{x}^A)$ -coordinates. From direct computations using (A.2), we see that

$$(A.3) \quad \begin{aligned} \underline{\Lambda}_{\rho\rho}^\rho &= \bar{g}_{\rho\rho} \underline{\rho} + \mathcal{O}(\underline{\rho}^2), & \underline{\Lambda}_{\rho a}^\rho &= \mathcal{O}(\underline{\rho}^2), \\ \underline{\Lambda}_{ab}^\rho &= -\bar{g}_{ab} \underline{\rho} + \mathcal{O}(\underline{\rho}^2), & \underline{\Lambda}_{\rho\rho}^a &= \mathcal{O}(\underline{\rho}^2), \\ \underline{\Lambda}_{\rho b}^a &= \mathring{g}^{ad} \bar{g}_{bd} \underline{\rho} + \mathcal{O}(\underline{\rho}^2), & \underline{\Lambda}_{bc}^a &= \mathcal{O}(1). \end{aligned}$$

<sup>21</sup>Much more precise comparisons hold; these can be found within the proof.

**A.1. Geodesic Coordinates.** First, note that we can formally extend  $\underline{h}$  to  $\rho \leq 0$  by dropping error terms and defining

$$(A.4) \quad \underline{h}|_{\rho \leq 0} := (1 + \rho^2 \bar{g}_{\rho\rho})d\rho^2 + (-d\underline{t}^2 + \dot{\bar{g}}_{AB}d\underline{x}^A d\underline{x}^B + \rho^2 \bar{g}_{ab}d\underline{x}^a d\underline{x}^b).$$

Observe that this extended  $\underline{h}$  is  $C^2$  in  $\rho$  and smooth in the remaining  $\underline{x}^a$ -coordinates. Thus, it makes sense to speak of derivatives of quantities “at  $\underline{\mathcal{I}} = \{\rho = 0\}$ ”.

Let  $\gamma$  be the family of  $\underline{h}$ -geodesics beginning on  $\rho = 0$  and satisfying the initial conditions  $\gamma'|_{\rho=0} = \partial_{\underline{\rho}}$ . Let  $\underline{\sigma}$  denote the affine parameter of these  $\gamma$ , with  $\underline{\sigma} = 0$  on  $\rho = 0$ . Furthermore, given coordinates  $(\underline{x}^a) = (\underline{t}, \underline{x}^A)$  on  $\rho = 0$ , we define coordinates  $(x^a) := (t, x^A)$  on the spacetime by transporting the  $\underline{x}^a$ 's along  $\gamma$ .

Consider now the coordinates  $(\underline{\sigma}, x^a)$ , and note that

$$\underline{D}_{\partial_{\underline{\sigma}}} \partial_{\underline{\sigma}} = 0, \quad \underline{h}(\partial_{\underline{\sigma}}, \partial_{\underline{\sigma}}) = \underline{h}(\partial_{\underline{\sigma}}, \partial_{\underline{\sigma}})|_{\rho=0} = 1,$$

where  $\underline{D}$  is the Levi-Civita connection for  $\underline{h}$ . Furthermore, we have

$$\partial_{\underline{\sigma}}[\underline{h}(\partial_{\underline{\sigma}}, \partial_{x^a})] = \underline{h}(\partial_{\underline{\sigma}}, D_{\partial_{\underline{\sigma}}} \partial_{x^a}) = \underline{h}(\partial_{\underline{\sigma}}, D_{\partial_{x^a}} \partial_{\underline{\sigma}}) = 0.$$

Thus, we can write  $\underline{h}$  as

$$(A.5) \quad \underline{h} = d\underline{\sigma}^2 + \underline{h}_{ab}d\underline{x}^a d\underline{x}^b.$$

In addition, we now restrict ourselves to  $t \in (T_-, T_+)$ , so that the geodesic  $\gamma$  emanating from each  $P \in \{\rho = 0\}$  with  $t(P) \in (T_-, T_+)$  exists for some uniform interval  $\underline{\sigma} \in (-\underline{\sigma}_0, \underline{\sigma}_0)$  while remaining in a compact subset of the extended  $\underline{\mathcal{M}}$ . In particular, in this region, all quantities under consideration will be bounded.

The remaining goal of this subsection is to compare vector fields in the  $(\underline{\sigma}, x^a)$ -coordinates with those in the  $(\rho, \underline{x}^a)$ -coordinates. For this, we define the coefficients

$$(A.6) \quad \partial_{\underline{\sigma}} := X^\rho \partial_{\underline{\rho}} + X^a \partial_{\underline{x}^a}, \quad \partial_{x^a} = A_a^\rho \partial_{\underline{\rho}} + A_a^b \partial_{\underline{x}^b}.$$

The goal, then, is to control the  $X^\alpha$ 's and  $A_a^\alpha$ 's.

**A.1.1. Bounds for the  $X^\alpha$ 's.** First, we note that

$$0 = \underline{D}_{\partial_{\underline{\sigma}}} \partial_{\underline{\sigma}} = \partial_{\underline{\sigma}}(X^\mu) \cdot \partial_\mu + X^\alpha X^\beta \underline{\Lambda}_{\alpha\beta}^\mu \cdot \partial_\mu,$$

which expands to a system of differential equations:

$$(A.7) \quad \begin{aligned} \partial_{\underline{\sigma}} X^\rho &= -(X^\rho)^2 [\bar{g}_{\rho\rho} \rho + \mathcal{Q}(\rho^2)] + X^a X^b [\bar{g}_{ab} \rho + \mathcal{Q}(\rho^2)] + 2X^\rho X^a \cdot \mathcal{Q}(\rho^2), \\ \partial_{\underline{\sigma}} X^c &= (X^\rho)^2 \cdot \mathcal{Q}(\rho^2) - 2X^\rho X^a \cdot [\dot{\bar{g}}^{cd} \bar{g}_{ad} \rho + \mathcal{Q}(\rho^2)] - X^a X^b \cdot \underline{\Lambda}_{ab}^c. \end{aligned}$$

Moreover, note that by (A.4), the equations (A.7) extend to  $\rho \leq 0$ , with the  $\mathcal{Q}(\rho^2)$ -terms vanishing on  $\rho \leq 0$ . The equations also imply that the  $X^\alpha$ 's are twice continuously differentiable in  $\rho$  in this extended region.

We can now determine the asymptotics of the  $X^\alpha$ 's at  $\underline{\sigma} = 0$ . By definition,

$$(A.8) \quad X^\rho|_{\underline{\sigma}=0} = 1, \quad X^a|_{\underline{\sigma}=0} = 0.$$

Furthermore, the evolution equations (A.7) imply that

$$(A.9) \quad \partial_{\underline{\sigma}}(X^\alpha)|_{\underline{\sigma}=0} = 0.$$

For second derivatives, we differentiate (A.7). Noting  $\partial_{\underline{\sigma}} \rho = X^\rho$ , we have

$$(A.10) \quad \begin{aligned} \partial_{\underline{\sigma}}^2(X^\rho)|_{\underline{\sigma}=0} &= -(X^\rho)^2 \cdot \bar{g}_{\rho\rho} X^\rho|_{\underline{\sigma}=0} + X^a X^b \cdot \bar{g}_{ab} X^\rho|_{\underline{\sigma}=0} = -\bar{g}_{\rho\rho}, \\ \partial_{\underline{\sigma}}^2(X^a)|_{\underline{\sigma}=0} &= -2X^\rho X^a \cdot \dot{\bar{g}}^{cd} \bar{g}_{ad} X^\rho|_{\underline{\sigma}=0} - \partial_{\underline{\sigma}}(X^a X^b \cdot \underline{\Lambda}_{ab}^c)|_{\underline{\sigma}=0} = 0. \end{aligned}$$

Since the  $\underline{x}^a$ - and  $x^a$ -coordinates coincide at  $\underline{\sigma} = 0$ , we can also use (A.7) to compute some higher derivatives at  $\underline{\sigma} = 0$ . Indeed, given any integer  $l > 0$  and arbitrary indices  $a_1, \dots, a_l$ , we have that

$$(A.11) \quad \begin{aligned} \partial_{x^{a_1}} \dots \partial_{x^{a_l}} X^\alpha|_{\underline{\sigma}=0} &= 0, \\ \partial_{\underline{\sigma}} \partial_{x^{a_1}} \dots \partial_{x^{a_l}} X^\alpha|_{\underline{\sigma}=0} &= 0, \\ \partial_{\underline{\sigma}}^2 \partial_{x^{a_1}} \dots \partial_{x^{a_l}} X^\rho|_{\underline{\sigma}=0} &= -\partial_{\underline{x}^{a_1}} \dots \partial_{\underline{x}^{a_l}} \bar{g}_{\rho\rho}, \\ \partial_{\underline{\sigma}}^2 \partial_{x^{a_1}} \dots \partial_{x^{a_l}} X^a|_{\underline{\sigma}=0} &= 0. \end{aligned}$$

Since we restrict ourselves to a relatively compact region in  $\underline{\mathcal{M}}$ , then applying Taylor's theorem along with (A.8)-(A.11) yields

$$(A.12) \quad \left| \partial_{x^{a_1}} \dots \partial_{x^{a_l}} \left( X^\rho - 1 + \frac{1}{2} \bar{g}_{\rho\rho} \underline{\sigma}^2 \right) \right| \lesssim_l \underline{\sigma}^3, \quad |\partial_{x^{a_1}} \dots \partial_{x^{a_l}} X^a| \lesssim_l \underline{\sigma}^3$$

for any  $l > 0$  and indices  $a_1, \dots, a_l$ . Since  $\partial_{\underline{\sigma}} \underline{\rho} = X^\rho$ , then integrating the first inequality in (A.12) with respect to  $\underline{\sigma}$  results in the estimate

$$(A.13) \quad \left| \partial_{x^{a_1}} \dots \partial_{x^{a_l}} \left( \underline{\rho} - \underline{\sigma} + \frac{1}{6} \bar{g}_{\rho\rho} \underline{\sigma}^3 \right) \right| \lesssim_l \underline{\sigma}^4.$$

Note also that (A.9)-(A.11) imply

$$(A.14) \quad \begin{aligned} \left| \partial_{x^{a_1}} \dots \partial_{x^{a_l}} \partial_{\underline{\sigma}} \left( X^\rho - 1 + \frac{1}{2} \bar{g}_{\rho\rho} \underline{\sigma}^2 \right) \right| &\lesssim_l \underline{\sigma}^2, & |\partial_{x^{a_1}} \dots \partial_{x^{a_l}} \partial_{\underline{\sigma}} X^c| &\lesssim_l \underline{\sigma}^2, \\ \left| \partial_{x^{a_1}} \dots \partial_{x^{a_l}} \partial_{\underline{\sigma}}^2 \left( X^\rho - 1 + \frac{1}{2} \bar{g}_{\rho\rho} \underline{\sigma}^2 \right) \right| &\lesssim_l \underline{\sigma}, & |\partial_{x^{a_1}} \dots \partial_{x^{a_l}} \partial_{\underline{\sigma}}^2 X^c| &\lesssim_l \underline{\sigma}. \end{aligned}$$

Next, by induction, we can take successive  $\underline{\sigma}$ -derivatives of (A.7) and control the left-hand side by the right-hand side (which is lower-order), using bounds already obtained in the previous iteration. (In particular, throughout these differentiations, we recall that  $\partial_{\underline{\sigma}}(\underline{\rho} - \underline{\sigma}) = X^\rho - 1$ .) From this process, we obtain

$$(A.15) \quad \begin{aligned} \left| \partial_{x^{a_1}} \dots \partial_{x^{a_l}} \partial_{\underline{\sigma}}^k \left( X^\rho - 1 + \frac{1}{2} \bar{g}_{\rho\rho} \underline{\sigma}^2 \right) \right| &\lesssim_l \underline{\sigma}^{3-k}, \\ |\partial_{x^{a_1}} \dots \partial_{x^{a_l}} \partial_{\underline{\sigma}}^k X^c| &\lesssim_l \underline{\sigma}^{3-k}, \end{aligned}$$

for each nonnegative integer  $k$ . From (A.15), we obtain the asymptotic bounds

$$(A.16) \quad \begin{aligned} X^\rho - 1 + \frac{1}{2} \bar{g}_{\rho\rho} \underline{\sigma}^2 &= \mathcal{O}_\sigma(\underline{\sigma}^3), \\ X^a &= \mathcal{O}_\sigma(\underline{\sigma}^3), \\ \underline{\rho} - \underline{\sigma} + \frac{1}{6} \bar{g}_{\rho\rho} \underline{\sigma}^3 &= \mathcal{O}_\sigma(\underline{\sigma}^4), \end{aligned}$$

where we define  $\mathcal{O}_\sigma(\zeta)$  as in Definition 2.4, but with respect to  $(\underline{\sigma}, x^a)$ -coordinates.

**A.1.2. Bounds for the  $A_a^\alpha$ 's.** Observe that the  $A_a^\alpha$ 's satisfy

$$(A.17) \quad A_a^\rho = \partial_{x^a} \underline{\rho}, \quad A_a^b = \partial_{x^a} \underline{x}^b, \quad \partial_{\underline{\sigma}} A_a^b = \partial_{x^a} X^b.$$

Thus, from (A.16) and (A.17), we conclude that

$$(A.18) \quad A_a^\rho + \frac{1}{6} \partial_{x^a} \bar{g}_{\rho\rho} \underline{\sigma}^3 = \mathcal{O}_\sigma(\underline{\sigma}^4), \quad A_a^b = \delta_a^b + \mathcal{O}_\sigma(\underline{\sigma}^4).$$

**A.2. The Metric Expansion.** From (A.16) and (A.18), we conclude that<sup>22</sup>

$$(A.19) \quad \begin{aligned} \underline{h}(\partial_{x^a}, \partial_{x^b}) &= A_a^c A_b^d \cdot \underline{h}(\partial_{\underline{x}^c}, \partial_{\underline{x}^d}) + A_a^c A_b^\rho \cdot \underline{h}(\partial_{\underline{x}^c}, \partial_\rho) \\ &\quad + A_a^\rho A_b^d \cdot \underline{h}(\partial_\rho, \partial_{\underline{x}^d}) + A_a^\rho A_b^\rho \cdot \underline{h}(\partial_\rho, \partial_\rho) \\ &= \underline{h}(\partial_{\underline{x}^a}, \partial_{\underline{x}^b}) + \mathcal{O}_\sigma(\underline{\sigma}^4). \end{aligned}$$

It then follows from (A.2), (A.5), and (A.19) that

$$(A.20) \quad g = \underline{\rho}^{-2} d\underline{\sigma}^2 + \underline{\rho}^{-2} [\underline{g}_{ab} + \bar{g}_{ab} \underline{\rho}^2 + \mathcal{O}_\sigma(\underline{\sigma}^3)] dx^a dx^b.$$

**A.2.1. The Radial Normalization.** We make one final change of variables  $\underline{\sigma} \mapsto \rho$ , satisfying that  $\rho = 0$  and  $\underline{\sigma} = 0$  coincide at  $\underline{\mathcal{I}}$ , and that

$$(A.21) \quad \frac{d\underline{\sigma}}{\underline{\rho}} = \frac{d\rho}{\rho}.$$

Rearranging the above and recalling the last part of (A.16) yields

$$(A.22) \quad \frac{d(\log \rho)}{d\underline{\sigma}} = \frac{1}{\underline{\rho}} = \frac{1}{\underline{\sigma}} + \frac{1}{6} \bar{g}_{\rho\rho} \underline{\sigma} + \mathcal{O}_\sigma(\underline{\sigma}^2),$$

and integrating (A.22) results in the relation

$$(A.23) \quad \rho = \underline{\sigma} \cdot e^{\frac{1}{12} \bar{g}_{\rho\rho} \underline{\sigma}^2 + \mathcal{O}_\sigma(\underline{\sigma}^3)} = \underline{\sigma} + \frac{1}{12} \bar{g}_{\rho\rho} \underline{\sigma}^3 + \mathcal{O}_\sigma(\underline{\sigma}^4).$$

Inverting the relation in (A.23) results in the expansion

$$(A.24) \quad \underline{\sigma} = \rho - \frac{1}{12} \bar{g}_{\rho\rho} \rho^3 + \mathcal{O}(\rho^4),$$

while (A.16) and (A.24) imply

$$(A.25) \quad \underline{\rho} = \rho - \frac{1}{4} \bar{g}_{\rho\rho} \rho^3 + \mathcal{O}(\rho^4).$$

Applying (A.21) and (A.25) to (A.20) yields the FG-aAdS expansion

$$(A.26) \quad g = \rho^{-2} d\rho^2 + \rho^{-2} \left[ \underline{g}_{ab} + \left( \bar{g}_{ab} + \frac{1}{2} \bar{g}_{\rho\rho} \underline{g}_{ab} \right) \rho^2 + \mathcal{O}(\rho^3) \right] dx^a dx^b.$$

Finally, to complete the proof, we set the isometry  $\Phi$  to be the map represented by the compositions of the coordinate transformations

$$(\underline{\rho}, \underline{x}^a) \mapsto (\underline{\sigma}, x^a) \mapsto (\rho, x^a).$$

## APPENDIX B. EINSTEIN-VACUUM SPACETIMES

Let  $(\mathcal{M}, g)$  denote an admissible FG-aAdS segment. In this appendix, we briefly elaborate on the case in which  $(\mathcal{M}, g)$  also satisfies the Einstein-vacuum equations<sup>23</sup>

$$(B.1) \quad \text{Ric}[g] = -\frac{n(n-1)}{2} g.$$

Recall (see [4], for instance) the following:

<sup>22</sup>From (A.16) and (A.18), we infer that the classes  $\underline{\mathcal{O}}(\zeta)$  and  $\mathcal{O}_\sigma(\zeta)$  coincide.

<sup>23</sup>We choose this particular normalization of the cosmological constant in equation (B.1) so that the AdS (and Kerr-AdS) metric has the expansion (2.8) at infinity.

**Proposition B.1.** *Suppose  $n \geq 3$ , and suppose also that  $(\mathcal{M}, g)$  satisfies (B.1). Then,  $-\bar{\mathfrak{g}}$  is precisely the Schouten tensor associated with  $(\mathcal{I}, \mathring{\mathfrak{g}})$ ,*

$$(B.2) \quad \mathring{P}_{ab} := \frac{1}{n-2} \left[ \mathring{Ric}_{ab} - \frac{1}{2(n-1)} \mathring{R} \cdot \mathring{\mathfrak{g}}_{ab} \right],$$

where  $\mathring{Ric}$  and  $\mathring{R}$  denote the Ricci and scalar curvatures on  $(\mathcal{I}, \mathring{\mathfrak{g}})$ .

**B.1. The Static Case.** We now further specialize to the case of static boundaries. More specifically, we assume our conformal boundary has the form

$$(B.3) \quad \mathcal{I} := (0, \pi T) \times \mathcal{S}, \quad \mathring{\mathfrak{g}} := -dt^2 + \gamma,$$

where  $(\mathcal{S}, \gamma)$  is an  $(n-1)$ -dimensional Riemannian manifold, with  $n \geq 3$ . Below, we show that in this setting, the pseudoconvexity property of Definition 3.12 can be directly connected to positivity of the Ricci curvature  $\mathcal{Ric}$  of  $(\mathcal{S}, \gamma)$ :

**Proposition B.2.** *Suppose  $n \geq 3$ , and suppose  $(\mathcal{M}, g)$  satisfies (B.1) and (B.3). If  $\mathcal{Ric} \geq (n-2)C$  uniformly on  $\mathcal{S}$  for some  $C > 0$ , then:*

(1) *For any  $c \in (0, C)$ , the following is uniformly positive definite:*

$$(B.4) \quad -\bar{\mathfrak{g}} - c^2 dt^2 + \left[ \frac{1}{2(n-1)(n-2)} \mathcal{R} - \frac{1}{2}(C^2 + c^2) \right] \cdot \mathring{\mathfrak{g}},$$

where  $\mathcal{R}$  denotes the scalar curvature of  $\gamma$ .

(2) *The pseudoconvexity property holds whenever  $T > C^{-1}$ , i.e., whenever  $(\mathcal{M}, g)$  has a time span of greater than  $C^{-1}\pi$ .*

Furthermore, if  $\mathcal{Ric} \leq 0$  at any point of  $\mathcal{S}$ , then the pseudoconvexity property cannot hold for  $(\mathcal{M}, g)$  for any value of  $T > 0$ .

*Proof.* Observe that direct computations yield

$$(B.5) \quad \begin{aligned} \mathring{P}_{tA} &\equiv 0, & \mathring{P}_{tt} &= \frac{1}{2(n-1)(n-2)} \mathcal{R}, \\ \mathring{P}_{AB} &= \frac{1}{n-2} \mathcal{Ric}_{AB} - \frac{1}{2(n-1)(n-2)} \mathcal{R} \cdot \gamma_{AB}, \end{aligned}$$

from which we obtain

$$(B.6) \quad \mathring{P} + \frac{1}{2(n-1)(n-2)} \mathcal{R} \cdot \mathring{\mathfrak{g}} = 0 dt^2 + \frac{1}{n-2} \mathcal{Ric}_{AB} dx^A dx^B,$$

from which (B.4) follows. In particular, (B.4) implies that Definition 3.12 is indeed satisfied whenever  $\xi = 0$  and  $T := c^{-1} > C^{-1}$ .

Finally, if  $\mathcal{Ric} \leq 0$  at some point  $Q \in \mathcal{S}$ , then we can see that at  $Q$ ,

$$-\bar{\mathfrak{g}} - \zeta \mathring{\mathfrak{g}} = \frac{1}{n-2} \mathcal{Ric}_{AB} dx^A dx^B - \left[ \frac{1}{2(n-1)(n-2)} \mathcal{R} + \zeta \right] \cdot \mathring{\mathfrak{g}}$$

cannot be made positive-definite for any  $\zeta$ , hence the pseudoconvexity property is violated. This completes the proof of the proposition.  $\square$

In particular, in the classical gravity setting  $n = 3$ , we have

$$(B.7) \quad \mathcal{Ric}_{AB} = \mathcal{K} \cdot \gamma_{AB},$$

where  $\mathcal{K}$  is the Gauss curvature of  $(\mathcal{S}, \gamma)$ . Proposition B.2 and (B.7) imply:

**Corollary B.3.** *Suppose  $n = 3$  and  $(\mathcal{M}, g)$  satisfies (B.1) and (B.3).*

- (1) If  $\mathcal{K} \geq C > 0$  uniformly on  $\mathcal{S}$ , then the pseudoconvexity property holds whenever  $T > C^{-1}$ , that is, when  $(\mathcal{M}, g)$  has time span greater than  $C^{-1}\pi$ .
- (2) If  $\mathcal{K} \leq 0$  at any point of  $\mathcal{S}$ , then the pseudoconvexity property cannot hold for  $(\mathcal{M}, g)$  for any value of  $T > 0$ .

Note for AdS (as well as Kerr-AdS) spacetime, we have  $\mathcal{K} \equiv 1$ , hence the pseudoconvex condition holds for time intervals of length greater than  $\pi$ .

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